

Mathematics

Class 12

Volume -I

By B. Harendran, M.Sc. B.Ed

Here is a new book to help you achieve 100% in your exams.

Be it test conducted in school as weekly, monthly, cumulative, or model exam, this book will assure you to get the best and one can achieve 100%.

This book has been designed and written by Mr. B Harendran, M.Sc., B.Ed, who is a Mathematician having taught in leading and best schools in Tamil Nadu, Kuwait, Bahrain .He has been teaching since 1978.

He had schooling in Delhi, collegiate educations in Chennai, had his stint in programming and software experience through Anna University, IIT.

Having gone through syllabus, books of NCERT, State Board and mathematics books followed by students, teachers in USA, UK, Singapore, This book is dedicated to students who will excel in Mathematics.

On finding the difficulty faced by students in answering, presenting answers in tests, exams, The author is bringing this book with the right guidance.

This book contains volume -1, and is meant for chapter concepts and problem solving. It begins with list of formulae, concept, definitions, and list of problems. The right solutions including any aliter answers are given . Some problems are let to students to try and practice.

Reader and students who will practice as per this book is the best judge of the content of the booklets.

Wishing all best performance, confide in Mathematics.

B Harendran M.Sc., B.Ed.

Presently working in National Model Sr.Sec School, Coimbatore.

Content:

Chapter -1: Relation and function

Chapter -2: Inverse trigonometric function

Chapter-3: Matrices

Chapter-4: Determinants



Chapter -1: Relation and function

Formulae used in this chapter:

- $A \times B = \{(a, b) : a \in A, b \in B\}$
- Relations are subsets of $A \times B$.
- In a relation element belonging to set B are *images* and elements belonging to A are pre-images.

We represent relations as (i) set of ordered pairs (ii) mapping using Venn Diagrams (iii) graphical points.

There are three types of relations: 1) reflexive relation (2) symmetric relation (3) transitive relation. There is a symmetric relations called *anti-symmetric*. *the identity relations are of this type*

if a relation is all the three types *reflexive, symmetric and transitive* relation then it is called ***equivalence relation***.

In Z there are partitions called equivalence class. [0],[1] are equivalence class of $|a - b|$ is divisible by 2.

Congruence modulo n : Let n be a non-zero integer, if n divides $|a - b|$ is it written as $a \equiv b \pmod{n}$

Number of relations:

Let $n(A)=m$, $n(B) = n$ then number of relations are 2^{mn} .

- Number of reflexive relations on $A \times A$ is 2^{n^2-n} .
- Number of symmetric relations on a set containing n elements is $2^{\binom{n^2+n}{2}}$. Number of
- transitive relations** is given by the following table:

<i>Number of elements</i>	<i>Number of transitive relations</i>
1	5
2	13
3	171
4	3994
N	Visit http://oeis.org/A006905

Functions:

$f: A \rightarrow B$, f is a function or mapping of all elements of A into B .

Here A is called domain, B is called co-domain and set of all images of every element of B is range.

Depending on range as subset of co-domain, or range is co-domain itself the functions are classified as INTO, ONTO functions.

If $\text{range} \subset B$, the function is called Into or injective function.

If $\text{range} = \text{co-domain}$, the function is called onto function or surjective function.

Functions are sub classified as One-One and Many -One.

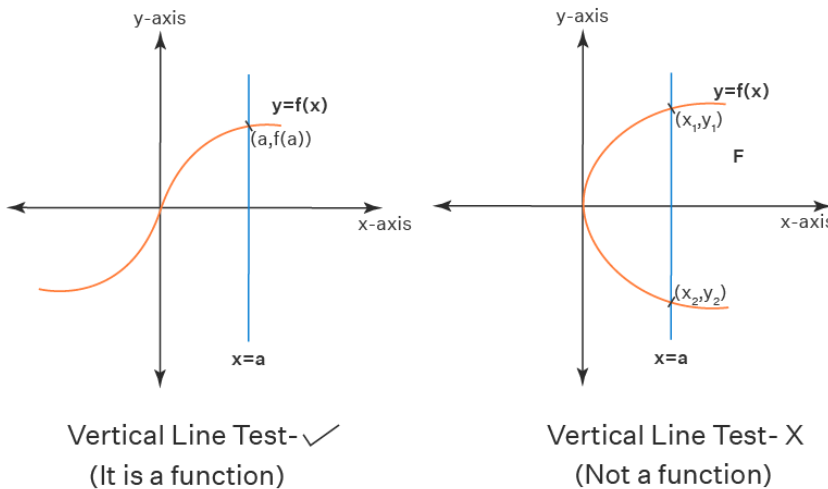
How to test a function?

Using the graph of the function, Vertical line test confirms a given relation of function or not.

What is vertical line test:

You can use the vertical line test on a graph to determine whether a relation is a function. **If it is impossible to draw a vertical line that intersects the graph more than once, then each x-value is paired with exactly one y-value.** So, the relation is a function.

The following diagram/graph explains vertical line test:



How to test a given function is Into and one-one:

List all elements of range and if it is a subset of B where $f: A \rightarrow B$ then function is *into*. **If range is the co-domain itself, then the function is ONTO**

- 1) take two elements for set A say x_1, x_2
- 2) Assume $f(x_1) = f(x_2)$
- 3) Simplify step 2 and arrive at either $x_1 = x_2$ or $x_1 \neq x_2$
- 4) If $x_1 = x_2$, then the given function is one-one function, otherwise it is many-one.

Inverse of a function:

Let f^{-1} be the inverse function of f , it obeys the following rules:

- 1) $f \circ f^{-1} = I$ (identity function).
- 2) $f^{-1} \circ f = I$

Algebra of functions:

Let f , g and h be real valued functions then

$$(f \pm g)(x) = f(x) \pm g(x)$$

$(fg)(x) = f(x)g(x)$...product of two functions

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \text{ where } g(x) \neq 0, \text{ division of two functions}$$

$(cf)(x) = c f(x)$ where c is a real constant.

$$(-f)(x) = -f(x)$$

If $f(-x) = f(x)$, $f(x)$ is called **even** function

If $f(-x) = -f(x)$ or $f(x) + f(-x) = 0$, $F(x)$ is called **odd** function.

There are many kinds of functions:

Identity function: $f(x) = x$, $x \in \mathbb{R}$

Algebraic function.:

- a) Polynomial function: $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_nx^n$, where $a_0, a_1, a_2, a_3, \dots, a_n$ are real constants.
- b) Reciprocal function: $f(x) = 1/x$, $x \neq 0$
- c) Rational function: $f(x) = \frac{p(x)}{q(x)}$, $q(x) \neq 0$ and $p(x)$, $q(x)$ are polynomial functions.
- d) Modulus function: $f(x) = |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$, $x \in \mathbb{R}$
- e) Signum function: $f(x) = \frac{|x|}{x}$, $x \neq 0$, $x \in \mathbb{R}$
- f) Greatest integer function: $f(x) = [x]$, $x \in \mathbb{R}$
- g) Floor function, ceiling function

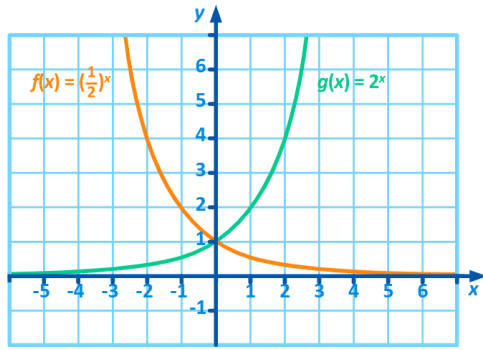
Trigonometric functions:

All functions where trigonometric ratios are used. $F(x) = \sin x$, $f(x) = \sin 2x + \cos 3x$ etc.,

Transcendental functions:

Exponential function:

Function defined by a relation in the form $f(x) = a^x$ where a is a strictly positive real number that is different from 1.



The function $f(x) = e^x$. The graph always passes through (0,1).

The function $f(x) = \log_a(x)$. Let $y = \log_a(x) \Rightarrow a^y = x$. Here a is called *base*.

When the base is e , $\log_e(x)$ is called Natural logarithm denoted as $\ln x$. When the base is 10, $\log_{10}x$ is called logarithm with base 10. (note: we use this logarithm for numerical calculations in Physics and Mathematics).

Most of Indian schools of state Board, CBSE, DBSE follow the syllabus for class 8 to 12 and books prescribed are given by NCERT.

RELATION and FUNCTION

Learning Objectives:

- (1) Relations and function, (2) kinds of function and their graphs. After learning these a concepts, and solving related problems, students are taught in class XII, Types of Relations, Types of functions, Inverse relations and inverting a given function if it exists.

Let us explore:

Relation:

Definition: consider the product a two finite sets A and B is $A \times B = \{ (x, y) : x \in A, y \in B \}$. (x, y) is called ordered pair. The SUBSETS of this $A \times B$ are all relations.

Notation $(x, y) \in A \times B$ mean x is related to y written as xRy .

R: Relation can be pure mathematical or human relation or relation between an object to another.

There is a special relation named as function.

What are image, pre-image?

When xRy y is the image of x and x is the pre-image of y .

When x has a unique image in R (relation) becomes function,

Here are some examples: [Algebraic & Venn-diagram]

$A = \{ 1, 2, 3, 4 \}$, $B = \{ 1, 2, 3, 4 \}$ NOW $A \times B = A \times A$ as $A=B$.

$A \times A = \{ (1,1), (1,2), (1, 3), (1,4) (2,1), (2, 2) (2,3), (2,4), (3,1), (3,2) (3,3), 3,4), (4,1) (4,2), 4,3) (4,4) \}$ consider the following sets :

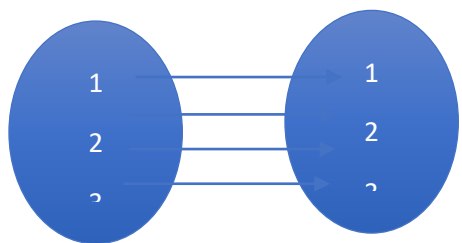
$R_1 = \{ (1,1), (2,2), (3,3), (4,4) \}$, $R_2 = \{ (1, 1) (1, 2) (1,3), (1,4) \}$

$R_3 = \{ (1, 2) (2,4) \}$ etc., $R_1 R_2 R_3$ are subsets of $A \times B$. Hence, they are Relation.

Let us form the table

Pre-Image	Image
1	1
2	2
3	3
4	4

Venn Diagram

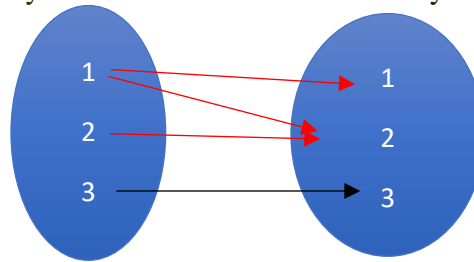


Reason:

Every element of A has a unique image (OR) every element of A has one-and-only image.

Arrowed diagrams image is called mapping.

R2 :



In set A element 1 has no unique image in B It can be only relation and not a function.

Note: In R2, every element of A does not have an image

Consider $R3 = \{(1,2), (2,4)\}$

Here if $A = \{1, 2\}$ and $B = \{1, 2, 3, 4\}$ R3 is a Relation as well as function *Reason:* If $A = \{1, 2\}$ then ever every element of A has a unique image in B.

Note: We can find $1 \times 2 = 2$ and $2 \times 2 = 4$.

hence if $x \in A$ than $2x \in B$.

or $x \rightarrow 2x$ (x is mapped to $2x$)

Thus: A function is clearly defined as a mapping from set A to B it to every element & A has one and image in B.

Also note: In R2 few elements of A does not have image; hence every element of A has no image; some elements are left out.

In R3: In set B, few elements are left out without any mapping, but still R3 is a function as element 1,2 has one and only image in B.

Other type of relations: *(Nonmathematical)*

"is a brother of", "lives in the same city", "Is a father of" are non- mathematical relations and treated as Relation.

Example: $A = \{\text{Ashok, Samuel, Raju}\}$ are preimage. Images are given by: Ashok is a brother of Raju. Ashok is a brother Raji

Ashok "lives in the same city as Samuel

Ashok "lives in the same city as Raji

Raja " in the same city, as Ashok.

Pre Image = $\{\text{Ashok, Raju}\}$ image = $\{\text{Samuel, Raji}\}$

Such relations may or may not form as a function.

Types of relation (class XII)

- (1) Reflexive relation
- (2) Symmetric relation
- (3) Transitive relation

Definitions.

- (1) In a relation R if $(a, a) \in R$ it is Reflexive
- 2) In a relation R if $(a, b), (b, a) \in R$ it is Symmetric
- 3) In a relation R if $(a, b), (b, c) \in R$ and $(a, c) \in R$ then R is Transitive.

More types of relations: (1) Empty relation (2) Universal relations. Empty relation: If no element of A is related to any element

$R = \phi$ (empty relation), $\phi \in R \times R$.

Universal relation: If each element of A is related to every element of $R = A \times A$ is UNIVERSAL.

Examples: $R = \{(a, b); a \text{ is a sister of } b, a \in A \text{ is a set of Boys in a school}\}$, R is empty Relation.

$R = \{(a, b) \text{ the difference between height of } a \text{ and } b \text{ is less than 3 meters}\}$ R is UNIVERSAL

EQUIVALENCE relation. If R is a relation and R is reflexive, symmetric and Transitive.

Solving Methods and correct explanation for problems related to type of relation.

Problems

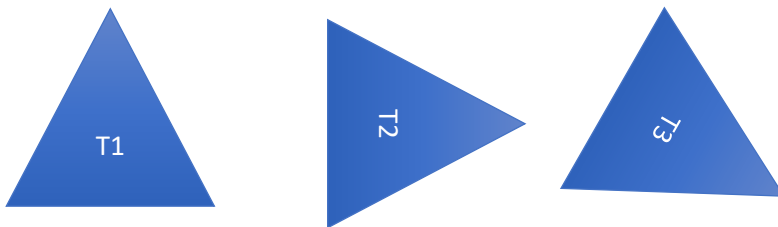
- 1) Let T be set of all triangles in a plane. $R = \{(T1, T2) \mid T1 \text{ is congruent to } T2\}$. Prove R is equivalence

Solution: 1: $(T1, T2) \in R$ and T1 is congruent to itself. Hence Reflexive

$(T1, T2) \in R \Leftrightarrow (T2, T1) \in R$. T1 is congruent to T2 then T2 is congruent to T2 also. So R is symmetric.

Let $(T1, T2)$ and $(T2, T3) \in R$;

T1 is congruent to T2, T2 is congruent to T3 implies T1 also congruent to T3



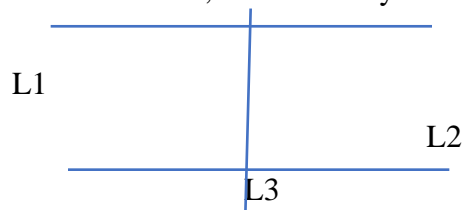
- 2) Let L be the set of all lines in a plane and R be the relation in L: $R = \{(L1, L2) : L1 \perp L2\}$ Is R equivalent.

Solution:

No

$(L1, L1) \notin R$ as L1 is not perpendicular to itself. Hence R is not reflexive.

$(L1, L2) \in R \Rightarrow L1 \perp L2 \Rightarrow L2 \perp L1$, hence R is Symmetric. Let $(L1, L2) \in R$, and $(L2, L3) \in R$ then $L1, L3 \notin R$. diagram:



Hence R is symmetric but neither reflexive nor transitive.

3) Let $A = \{1, 2, 3\}$ and $R = \{(1,1), (2,2), (3,3), (1,2), (2,3)\}$

Note: To prove reflexive $(a, a) \in R$.

Here $(1, 1), (2, 2), (3, 3) \in R$, hence R is reflexive. b) if one of the ordered pair $(1, 1), (2, 2), (3, 3)$ then R is not reflexive.

R is **symmetric** if (a, b) and $(b, a) \in R$.

In R $(1,2) \in R$ but $(2,1) \notin R$, $(2,3) \in R$ but $(3,2) \notin R$ hence R is not symmetric.

R is Transitive: If $(a, b), (b, c) \in R \Rightarrow (a, c) \in R$

Here $(1, 2), (2, 3) \in R$ but $(1, 3) \notin R$, $\therefore R$ is not transitive.

Hence R is symmetric but neither reflexive nor transitive.

Also, R is not equivalence.

(4) Show that the relation R in the set Z of integers given by $R = \{(a, b) : 2 \text{ divides } a-b\}$ is an equivalent relation.

Solution:

If $(a, a) \in R$, R is reflexive, 2 divides $a - a = 0$; R is reflexive.

If (a, b) and $(b, a) \in R$ then R is symmetric. Here 2 divides $a-b \Rightarrow 2$ divides $b-a$ also.

Let $a-b = 2\lambda$, then $b-a = 2(-\lambda)$, hence R is symmetric.

Proof of transitivity:

Let $(a, b) \in R \Rightarrow a-b = 2\lambda \dots(1)$

Let $(b, c) \in R \Rightarrow b-c = 2\mu \dots(2)$

$$(1) + (2) \Rightarrow (a - b) + (b - c) = (a - c) = 2(\lambda + \mu) = 2k$$

Therefore $(a, c) \in R$, hence R is transitive.

(5)

Consider $E = \{\text{set of all even integers}\}$, $O = \{\text{set of all odd integers}\}$

$R_1 = \{(0, \pm 2), (0, \pm 4), \dots, (0, \pm n), n \in \mathbb{Z}\}$ is equivalence using 2 divides $a-b$.

$R_2 = \{(1, \pm 1), (1, \pm 3), (1, \pm 5), \dots, (1, 2n+1)\}$ using 2 divides $a-b$.

Equivalent classes:

Definition:

Congruence modulo n:

Let n be a non-zero integer. The numbers $a, b \in \mathbb{Z}$. If n divides $(a-b)$ then is called congruence modulo n . It is written as $a \equiv b \pmod{n}$.

If 2 divides $a-b$. then $a \equiv b \pmod{2}$

Examples: $a=7, b=5$ then $|7-5| = 2$ or $|5-7| = 2$ which is divisible by 2. Therefore, it is written as $7 \equiv 5 \pmod{2}$ or $5 \equiv 7 \pmod{2}$. In \mathbb{Z} divide any number by 2 either the remainder is 0 or 1.

Thus there is a class denoted as $[0] = \{\dots, -4, -2, 0, 2, 4, \dots\}$ and $[1] = \{\dots, -5, -3, -1, 1, 3, 5, \dots\}$ such that $[0] \cup [1] = \mathbb{Z}$.

$[0], [1]$ are equivalent classes.

When $a \equiv b \pmod{3}$ it has $[0], [1], [2]$ equivalent classes.

Thus, possible remainders when an integer is divided by 2 is 0, 1

Possible remainders when an integer is divided by 3 is 0, 1, and 2

Possible remainders when an integer is divided by n is $[0], [1], [2], [3] \dots [n-1]$. There are n equivalent classes.

The following sets

$A_1 = [3r], A_2 = [3r+1], A_3 = [3r+2]$ are equivalent classes when a number is divided by 3. R is

$A_1 \cup A_2 \cup A_3 = \mathbb{Z}$ (set of integers).

Solved Problems:

- 1) In a set $A = \{1, 2, 3, 4, 5, 6, 7\}$ $R = \{(a, b) : \text{both } a \text{ and } b \text{ are either even or odd}\}$ Test R is an equivalence relation or not?
- 2) $A = \{1, 2, 3, 4, 5, 6, 7, \dots, 13, 14\}$ R is defined by $R = \{(x, y) : y = 3x\}$, is R equivalence relation?
- 3) In set $N = \{1, 2, 3, \dots\}$ $R = \{(x, y) : y = x + 5 \text{ and } x < 4\}$, is R equivalence?
- 4) In set $A = \{1, 2, 3, 4, 5, 6\}$, $R = \{(x, y) : y \text{ is divisible by } x\}$, discuss the type of relation R .
- 5) In \mathbb{Z} , $R = \{(x, y) : x - y \text{ is an integer}\}$ Is R transitive?
- 6) In a cryptography coding it is found every letter of the word "LET US WIN" mapped to a letter three places later it. Show the mapping, is the relation equivalence.
- 7) Let A be the set of children and elders of a family. R is such that $R = \{(a, b) : a \text{ is sister of } b\}$. Discuss the relation as equivalence or not.
- 8) Let $X = \{1, 2, 3, 4\}$ and $R = \{(1, 1), (2, 2), (3, 3), \dots, (n, n)\}$ is R equivalence.
- 9) Let $S = \{1, 2, 3\}$ and $\rho = \{(1, 1), (2, 2), (1, 2), (1, 3), (3, 1)\}$
 - a. Is ρ reflexive?
 - b. Write minimum set of ordered pairs to be included to make ρ reflexive?
 - c. Is ρ symmetric?
 - d. Write minimum set of ordered pairs to be deleted to make ρ symmetric?
 - e. Is ρ transitive?
 - f. Write minimum set of ordered pairs to be included to make ρ transitive?
 - g. Is ρ equivalence?
 - h. Is Write minimum set of ordered pairs to be included to make ρ equivalence?
- 10) Let $A = \{0, 1, 2, 3\}$ constructing relation on A for the following type :
 - a. Not reflexive, not symmetric, not transitive
 - b. Not reflexive, not symmetric, transitive
 - c. Not reflexive, symmetric, transitive
 - d. reflexive, not symmetric, not transitive
 - e. reflexive, not symmetric, transitive
 - f. reflexive, not symmetric, transitive
 - g. equivalence

- 11) In set Z of integers, mR_n if $m-n$ is a multiple of 12. Prove R is equivalence.
- 12) How many relations are there from set A to B , $n(A)=m$, $n(B)=n$.
- 13) How many relations are there from set A to A ?
- 14) How many reflexive relations are there from A to A if $n(A)=n$.
- 15) How many symmetric relations are there on a set containing n elements?
- 16) How many equivalence relations are there in A with 3 elements?
- 17) If R is a relation from A to B , what is the domain and range of the relation?

MCQ type questions:

- 1) Let R be a relation in the set $\{1, 2, 3, 4\}$ given by $R = \{(1, 2), (2, 2), (1, 1), (4, 4), (1, 3), (3, 3), (3, 2)\}$ then (a) R is reflexive and symmetric but not transitive. (b) R is reflexive and transitive but not symmetric. (c) R is symmetric and transitive but not reflexive. (d) R is equivalence
- 2) Let R be the relation in the set N given by $R = \{(a, b) : a = b - 2, b > 6\}$ then (a) $(2, 4) \in R$ (b) $(3, 8) \in R$ (c) $(6, 8) \in R$ (d) $(8, 7) \in R$.
- 3) Let $X = \{a, b, c, d\}$ and $R = \{(a, a), (b, b), (c, c)\}$ write down the minimum number of ordered pairs to be included to R to make it (a) reflexive (b) Symmetric (c) Transitive (d) equivalence
- 4) Let $A = \{1, 2, 3\}$, then number of relations containing $(1, 2)$ and $(1, 3)$ are reflexive and symmetric but not transitive is (a) 1 (b) 2 (c) 3 (d) 4
- 5) Let $A = \{1, 2, 3\}$ the number of equivalence relations containing $(1, 2)$ is (a) 1 (b) 2 (c) 3 (d) 5
- 6) What type of relation is "*less than*" in the set of real numbers? (a) only symmetric (b) only transitive (c) only reflexive (d) none
- 7) Consider the non-empty set consisting of children in a family and a relation R defined as aR_b if a is *a brother of* b , then R is (a) symmetric but not transitive (b) transitive but not symmetric (c) neither symmetric nor transitive (d) both symmetric and transitive
- 8) The maximum number of equivalence relations on the set $A = \{1, 2\}$ is (a) 1 (b) 2 (c) 3 (d) 4
- 9) If a relation R on the set $\{1, 2, 3\}$ be defined as $R = \{(1, 2)\}$, then R is (a) reflexive (b) transitive (c) symmetric (d) none
- 10) A relation R in real numbers set is aRb if $a \geq b$, then R is (a) (a) an equivalence (b) reflexive, transitive but not symmetric (c) neither transitive nor reflexive but symmetric. (d) symmetric, transitive but not reflexive.
- 11) The relation R is defined on the set of natural numbers as $R = \{(a, b) : a = 2b\}$ then R^{-1} is given by (a) $\{(2, 4), (4, 2), (6, 3), \dots\}$ (b) $\{(1, 2), (2, 4), (3, 6), \dots\}$ (c) R^{-1} is not defined (d) none of these
- 12) Which of the following relations on R is an equivalence relation? (a) $aRb \Leftrightarrow |a| = |b|$ (b) $aRb \Leftrightarrow a \geq b$ (c) $aRb \Leftrightarrow a$ divides b (d) $aRb \Leftrightarrow a < b$
- 13) Let R be the relation on the set of natural numbers denoted by $nRm \Leftrightarrow n$ is *a factor of* m . R is (a) reflexive and symmetric (b) transitive and symmetric (c) equivalence (d) reflexive, transitive but not symmetric.
- 14) Let R be the relation in the set of all *straight lines in a plane* such that l_1, l_2 are straight lines and $l_1 \perp l_2$, then R is (a) symmetric (b) reflexive (c) transitive (d) an equivalence
- 15) Let $A = \{2, 3, 4, 5, \dots, 17, 18\}$. "*≈*" is an *equivalence* relation in $A \times A$, defined by $(a, b) R (c, d)$ such iff $ad = bc$, the number of ordered pair of the equivalence class of $(3, 2)$ is (a) 4 (b) 5 (c) 6 (d) 7
- 16) The relation in $N \times N$ is $(a, b) R (c, d) \Leftrightarrow a + d = b + c$. R is (a) reflexive but not symmetric (b) reflexive and transitive but not symmetric (c) an equivalence (d) none of these.

- 17) A relation ρ from C to R is defined as xRy is defined by $y = |x|$. Which one is correct? (a) $(2+3i)R_{13}$
 (b) $3R_{(-3)}$ (c) $(1+i)R_2$ (d) iR_1
- 18) Let R be the relation on N defined by $x + 2y = 8$. The domain of R is (a) $\{2,4,8\}$ (b) $\{2,4,6,8\}$ (c) $\{2, 4, 6\}$ (d) $\{1,2,3,4\}$
- 19) R is a relation from $\{11, 12, 13\}$ to $\{8, 9,10\}$ defined by $y = x - 3$, then R^{-1} is (a) $\{(8,11), (9,12), (10,13)\}$ (b) $\{(11,8),(13,10)\}$ (c) $\{(10,13),(8,11),(8,10)\}$ (d) none of these.
- 20) Let $R = \{(a, a), (b, b), (c, c), (a, b)\}$ be a relation on $A = \{a, b, c\}$. R is (a) identity relation (b) reflexive relation (c) symmetric relation (d) equivalence

Short Answer type questions:

- Show that the relation R in a set A of points in a plane given by $R = \{(P, Q): \text{distance of } P \text{ from origin is same as the distance of } Q \text{ from origin}\}$. (a) prove R is equivalence (b) Show that set of all points $P \neq (0, 0)$ is a circle passing through P with origin as center.
- Show that the relation R defined on the set of all triangles as $R = \{(T_1, T_2): T_1 \text{ is similar to } T_2\}$ is equivalence. Consider three right triangles T_1 with sides $(3, 4, 5)$, T_2 with sides $(5, 12, 13)$ and T_3 with sides $(6,8,10)$; which triangles are related.
- Show that the relation R defined in the set of *polygons* as $R = \{P_1, P_2\}: P_1 \text{ has same number of sides as that of } P_2\}$ is equivalence relation.
- Let L be the set of all lines in XY plane and R be the relation defined as $R = \{(L_1, L_2): L_1 \text{ is parallel to } L_2\}$. Show R is equivalence. Find the set of all lines related to the line $y = 2x + 4$.

Long Answer type:

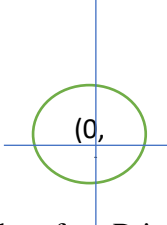
- If R_1 and R_2 are equivalence relations in a set A , show that $R_1 \cap R_2$ is also an equivalence relation.
- Let R be a relation on the set A of ordered pairs of positive integers defined by $(x, y)R(u, v)$ iff $xv = yu$. Show that R is equivalence.
- Let $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ The relation $\rho_1 = \{(x, y): x - y \text{ is divisible by } 3\}$ and $\rho_2 = \{(x, y): \{x, y\} \subset \{1, 4, 7\} \text{ or } \{x, y\} \subset \{2, 5, 8\} \text{ or } \{x, y\} \subset \{3, 6, 9\}\}$. Show that $\rho_1 = \rho_2$.
- Lines $y = m_1x + c_1, y = m_2x + c_2, R = \{\text{parallel lines in a plane}\}$. What are the conditions for R to be equivalence?
- Set $A = \{1, 2, 3\}$, how many relations are there in $A \times A$. List all of them
- Set $A = \{1, 2, 3\}$ $B = \{a, b, c\}$ form all equivalence relation in $A \times B$. Write the conditions to justify your answers.
- set $A = \{1, 2, 3\}$; how many reflective relations are there?
- Set $A = \{1, 2, 3\}$ how many symmetric relations are there? Also if $n(A) = 4$ find the number of symmetric relations?
- What are the equivalence classes of the relation $R = \{(a, b): a \equiv b \pmod{3}\}$
- How many equivalence relations are there in $A \times A$ when $n(A) = 4$?

Answers:

MCQ:

- 1) (a) (2) a, c (3) add (d, d) in (a) , add (a,b),(b,a) in (b) , add (a,b), (b, a) in (c) and add (d, d) in (d)
(4) (a) (5) 1 (6) (d) (7) (b) (8) 5 (9) (b) (10) b (11) a (12) a (13) d (14) a (15) a, b (16)
equivalence relation (17) d (18) c (19) a (20) b

Short Answers:

- 1) $R = \{(P, Q) : OP = OQ, \text{ where } P, Q \text{ are points in a cartesian plane}\}$
- a. $(P, P) \Rightarrow OP = OP$ hence R is reflexive
- b. $(P, Q) \Rightarrow OP = OQ \Rightarrow OQ = OP \Rightarrow (Q, P) \in R$, hence R is symmetric.
- c. Let $(P, Q), (Q, R) \in R$ then $OP = OQ = OR \Leftrightarrow OP = OR$, hence $(P, R) \in R$, therefore R is transitive.
- d. If $P \neq (0, 0)$ then P, Q, R will lie on a circle with center at origin (0, 0) and radius OP. see fig.
- 
- 2) $R = (T_1, T_2) : T_1$ is similar to T_2
- a. $(T_1, T_1) \in R$ as T_1 is a triangle similar to itself.
- b. $(T_1, T_2) \in R \Rightarrow (T_2, T_1) \in R$, hence symmetric.
- c. $(T_1, T_2), (T_2, T_3) \in R$, then $(T_1, T_3) \in R$ because $T_1 \approx T_2 \approx T_3 \Rightarrow T_1 \approx T_3$.
- Hence R is equivalence. Triangle with sides 3,4,5 is a right angled triangle, similarly 5, 12, 13 and 6,8,10 are Pythagorean triplets, hence all such triangles will be right angled. Hence they are similar to each other, Hence they belong to R.

3) Try yourself

4) Try yourself

Long Answer type:

1. Rewrite the answers given in example 41 (NCERT textbook)
2. Rewrite the answers given in example 42 (NCERT textbook)
3. Rewrite the answers given in example 43 (NCERT textbook)
4. $n(A) = 3; n(A \times B) = 9$, number of relations is $2^9 = 512$
5. try
6. set $A = \{1, 2, 3\}$; how many reflective relations are there?
7. Formula : hence answer is $2^{9-3} = 2^6 = 64$
8. Formula : answer $2^{\binom{3^2+3}{2}} = 2^6 = 64$
9. $R = \{(a, b) : a \equiv b \pmod{3}\}$
 - a. $(a, a) \in R$ as $a \equiv a \pmod{3}$, hence R is reflexive.
 - b. $(a, b) \in R \Rightarrow (b, a) \in R$ as $a \equiv b \pmod{3} \Rightarrow b \equiv a \pmod{3}$, hence R is symmetric.
 - c. $(a, b), (b, c) \in R \Rightarrow a \equiv b \pmod{3}, b \equiv c \pmod{3}$, adding them we get $a \equiv c \pmod{3}$.

10. Number of equivalence relation from A to A when $n(A) = 4$ is 3994.

11. Let $A = \{x \in \mathbb{Z} : 0 \leq x \leq 12\}$.

Show that $R = \{(a, b) : a, b \in A; |a - b| \text{ is divisible by } 4\}$ is an equivalence relation. Find the set of all elements related to 1. Also write the equivalence class [2]. (C.B.S.E 2018)

Solution:

We have:

$$R = \{(a, b) : a, b \in A; |a - b| \text{ is divisible by } 4\}.$$

(1) **Reflexive:** For any $a \in A$,

$$\therefore (a, a) \in R.$$

$$|a - a| = 0, \text{ which is divisible by } 4.$$

Thus, R is reflexive.

Symmetric:

Let $(a, b) \in R$

$$\Rightarrow |a - b| \text{ is divisible by } 4$$

$$\Rightarrow |b - a| \text{ is divisible by } 4$$

Thus, R is symmetric.

Transitive: Let $(a, b) \in R$ and $(b, c) \in R$

$$\Rightarrow |a - b| \text{ is divisible by } 4 \text{ and } |b - c| \text{ is divisible by } 4$$

$$\Rightarrow |a - b| = 4\lambda$$

$$\Rightarrow a - b = \pm 4\lambda \dots\dots\dots(1)$$

$$\text{and } |b - c| = 4\mu$$

$$\Rightarrow b - c = \pm 4\mu \dots\dots\dots(2)$$

Adding (1) and (2),

$$(a-b) + (b-c) = \pm 4(\lambda + \mu)$$

$$\Rightarrow a - c = \pm 4(\lambda + \mu)$$

$$\Rightarrow (a, c) \in R.$$

Thus, R is transitive.

Now, R is reflexive, symmetric and transitive.

Hence, R is an equivalence relation.

(ii) Let 'x' be an element of A such that $(x, 1) \in R$

$\Rightarrow |x - 1|$ is divisible by 4

$\Rightarrow x - 1 = 0, 4, 8, 12, \dots$

$\Rightarrow x = 1, 5, 9, 13, \dots$

Hence, the set of all elements of A which are related to 1 is $\{1, 5, 9\}$.

(iii) Let $(x, 2) \in R$.

Thus $|x - 2| = 4k$, where $k \leq 3$.

$\therefore x = 2, 6, 10$.

Hence, equivalence class $[2] = \{2, 6, 10\}$.

12. Let N denote the set of all natural numbers and R be the relation on $N \times N$ defined by : $(a, b) R(c, d)$ is $ad(b + c) = bc(a + d)$.

Show that R is an equivalence relation. (C.B.S.E. 2015)

Solution:

We have : $(a, b) R(c, d)$

$\Rightarrow ad(b + c) = bc(a + d)$ on N.

(i) $\{a, b\} R(a, b)$

$\Rightarrow ab(b + a) = ba(a + b)$

$\Rightarrow ab(a + b) = ab(a + b)$,

which is true.

Thus R is reflexive.

(ii) $(a, b) R(c, d)$

$\Rightarrow ad(b + c) = bc(a + d)$

$\Rightarrow bc(a + d) = ad(b + c)$

$\Rightarrow cb(d + a) = da(c + b)$

[$\because bc = cb$ and $a + d = d + a$;

etc. $\forall a, b, c, d \in N$]

$\Rightarrow (cb) R(a, b)$.

Thus R is symmetric

(iii) Let $(a, b) R(c, d)$ and $(c, d) R(e, f)$

$\therefore ad(b + c) = bc(a + d)$

and $cf(d + e) = de(c + f)$

$$\Rightarrow \frac{b+c}{bc} = \frac{a+d}{ad}$$

$$\text{and } \frac{d+e}{de} = \frac{c+f}{cf}$$

$$\Rightarrow \frac{1}{c} + \frac{1}{b} = \frac{1}{d} + \frac{1}{a}$$

$$\text{and } \frac{1}{e} + \frac{1}{d} = \frac{1}{f} + \frac{1}{c}$$

$$\Rightarrow \frac{1}{c} - \frac{1}{d} = \frac{1}{a} - \frac{1}{b}$$

$$\text{and } \frac{1}{c} - \frac{1}{d} = \frac{1}{e} - \frac{1}{f}$$

$$\Rightarrow \frac{1}{a} - \frac{1}{b} = \frac{1}{e} - \frac{1}{f}$$

$$\Rightarrow \frac{1}{a} + \frac{1}{f} = \frac{1}{b} + \frac{1}{e}$$

$$\Rightarrow be(a+f) = af(b+e)$$

$$\Rightarrow af(b+e) = be(a+f)$$

$$\Rightarrow (a, b) R (e, f).$$

Thus R is transitive.

Hence, R is an equivalence relation.

All about functions

From relation to function, we define function.

Let A and B be two sets. A relation from A to B, $f \subset A \times B$, is called function satisfying the condition:
 (a) for all $a \in A$, there is an element $b \in B$ such that $(a, b) \in f$. (b) if $(a, b) \in f$ and $(a, c) \in f$ then $b = c$.

In other words every element of A has one only image in B. OR no two ordered pair of a f has the same first element.

Domain and Range: The set A is called **domain** and all elements $b \in B$ such that $f(a)=b$ is called **range**.

Co domain: The whole set B is called co domain.

There are 2 cases for range:

Case 1: If range set is a subset of B, then the function is called Into function (also named as *injective* function).

Case 2: If range is equal to set B, then the function is called Onto function. (also named as *surjective* function)

Notations: $f: A \rightarrow B$ read as f is a mapping of all elements of A into B.

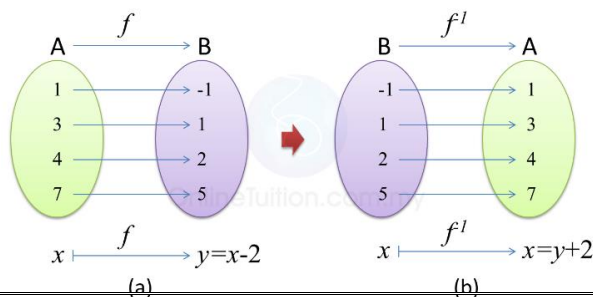
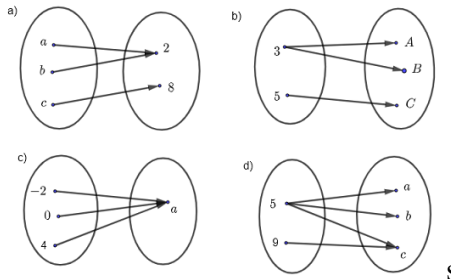
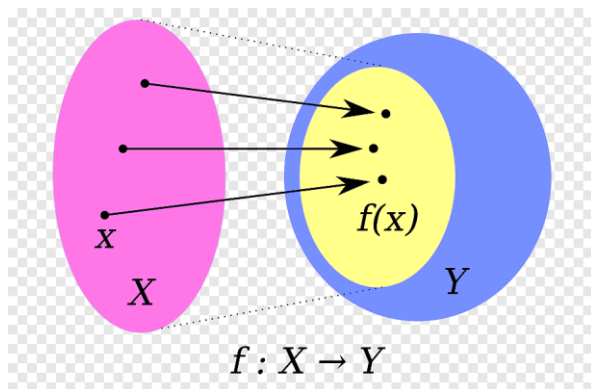
Let $a \in A, b \in A$ the function is written as $f(a) = b$.

If co domain set $B \subset \mathbb{R}$ (real numbers), the function is called real valued function.

The following are types of function:

- 1) One – One function (2) many one function
- 2) Any functions which is such that range \subset co domain they are called into function
- 3) Any functions which is such that range = co domain they are called onto function, a one-one onto function is called Bijective function.

Venn Diagrams for all types of functions:



$F : X \rightarrow Y$, X is domain, Y is co-domain. $F(x) \subset Y$, hence an into (injective) function. (a) is a Many one **onto** function (b) not a function (c) Many one onto (d) not a function.

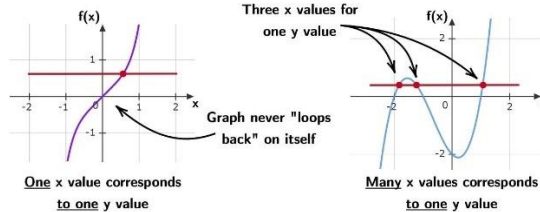
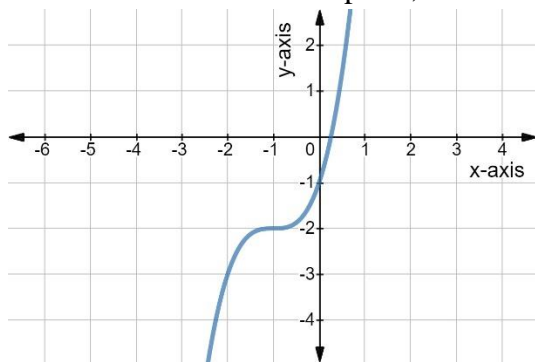
Condition on cardinal numbers of domain and co-domain:

Let A and B be two sets with m and n elements.

- I. There is no one-to-one function from A to B if $m > n$.
- II. If there is a one-to-one function from A to B then $m \leq n$.
- III. There is no onto function if $m < n$.
- IV. There is an onto function from A to B then $m \geq n$.
- V. There is a bijection from A to B , if and only if $m = n$.
- VI. There is no bijection from A to B if and only if $m \neq n$.

How to test a one – one function:

- (a) By algebraic method: Consider two elements a, b in the domain and assume $f(a) = f(b)$. After simplifying if $a=b$, then the function is one-one.
- (b) In case if $a \neq b$, then function is many one.
- (c) Graphical test: using horizontal line test. If a horizontal line (parallel to x -axis) meets the graph of a function at more than one point, the function is not one and many one. See fig.



Under what condition does an inverse function exists?

If a function is *bijjective* (one-one and onto) the inverse function exists.

How to find inverse function?

a) There are 3 methods

Method 1:

- 1) Replace every x by y and y by x .
- 2) Solve the equation from step 2 for y .
- 3) Replace y with $f^{-1}(x)$.
- 4) Verify that $(f \circ f^{-1})(x) = x$, the identity function.

Method 2:

Assume $y=f(x)$,
Express x in terms of y
Replace y by x we get $f^{-1}(x)$.

Method 3:

Draw the graph of the function (use www.desmos.com graph calculator). Find the reflection of the

graph about $y=x$ (straight line passing through origin). The reflection is inverse function.

Questions:

MCQ type:

Question 1.

The function $f : A \rightarrow B$ defined by $f(x) = 4x + 7, x \in \mathbb{R}$ is

(a) one-one (b) Many-one (c) Odd (d) Even

Answer:

(a) one-one

Question 2.

The smallest integer function $f(x) = [x]$ is

(a) One-one (b) Many-one (c) Both (a) & (b) (d) None of these Answer:

(b) Many-one

Question 3.

The function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = 3 - 4x$ is

(a) Onto (b) Not onto (c) None one-one (d) None of these

Answer:

(a) Onto

Question 4.

If $f : \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = 2x + 3$ and $g(x) = x^2 + 7$, then the value of x for which $f(g(x)) = 25$ is

(a) ± 1 (b) ± 2 (c) ± 3 (d) ± 4

Answer: (b) ± 2

Question 5.

If $f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \forall x_1, x_2 \in A$ then the function $f: A \rightarrow B$ is

(a) one-one (b) one-one onto (c) onto (d) many one

Answer

Question 6.

If $F : \mathbb{R} \rightarrow \mathbb{R}$ such that $f(x) = 5x + 4$ then which of the following is equal to $f^{-1}(x)$.

(a) $(x-5)/4$

(b) $x-y/5$

(c) $x-4/5$

(d) $4x - 5$

Question 7. Let the function 'f' be defined by $f(x) = 5x^2 + 2 \forall x \in \mathbb{R}$, then 'f' is

(a) onto function (b) one-one, onto function (c) one-one, into function (d) many-one into function.

Question 8:

The function $f(x) = x^2$, is a bijection if domain and co-domain are given by (a) \mathbb{R}, \mathbb{R} (b) $\mathbb{R}, (0, \infty)$ (c) $(0, \infty), \mathbb{R}$ (d) $[0, \infty), [0, \infty)$

Question 9

The function $f: [0, 2\pi] \rightarrow [-1, 1], f(x) = \sin x$ is (a) one-one (b) onto (c) bijection (d) cannot be defined

Question 10

Let $X = \{1, 2, 3, 4\}$ and $Y = \{a, b, c, d\}$ and $f = \{(1, a), (4, b), (2, c), (3, d), (2, d)\}$ is (a) one-one (b) onto (c) not one-one (d) not a function

Question 11

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined as $f(x) = 1 - |x|$. The range is (a) \mathbb{R} (b) $(1, \infty)$ (c) $(-1, \infty)$ (d) $(-\infty, 1]$

Question 12

The range of the function $f(x) = \frac{1}{1-2\sin x}$ is (a) $(-\infty, -1) \cup (1/3, \infty)$ (b) $(-1, 1/3)$ (c) $[0, 1)$ (d) $(0, 1)$

Question 13

Discuss the number of functions from set A to B?

If a set A has m elements and set B has n elements, then the number of functions possible from A to B is n^m . For example, if set $A = \{3, 4, 5\}$, $B = \{a, b\}$. If a set A has m elements and set B has n elements, then the number of onto functions from A to B = $n^m - {}^n C_1(n-1)^m + {}^n C_2(n-2)^m - {}^n C_3(n-3)^m + \dots - {}^n C_{n-1}(1)^m$.

Question 14

The number of onto functions (surjective functions) from set $X = \{1, 2, 3, 4\}$ to set $Y = \{a, b, c\}$ is:

(A) 36

(B) 64

(C) 81

(D) 72

Solution: Using $m = 4$ and $n = 3$, the number of onto functions is:

$$3^4 - {}^3 C_1(2)^4 + {}^3 C_2(1)^4 = 36$$

Question 15:

How many different functions are there from a set with 10 elements with the following number of elements? a) There are $2^{10} = 1024$ functions from 10 elements to 2 elements; b) $3^{10} = 59049$ from 10 elements to 3; c) $4^{10} = 1048576$ functions from 10 elements to 4 elements; and $5^{10} = 9,765,625$ functions from 10 elements to 5.

Question 16:

How many one-to-one functions are there from a set with five elements to a set with six elements?

Answer: 0.

Question 17:

How many injective functions are there from $\{1, 2, 3\}$ to $\{1, 2, 3, 4, 5\}$? Solution. Let f be such a function. Then $f(1)$ can take 5 values, $f(2)$ can then take only 4 values and $f(3)$ - only 3. Hence the total number of functions is $5 \times 4 \times 3 = 60$.

Short Answer type questions (Answered)

- 1) Check whether $f: \mathbb{N} \rightarrow \mathbb{N}$ defined by $f(n) = n + 2$?

Answer: Let $f(n) = f(m)$, then $n + 2 = m + 2 \Rightarrow n = m$. Hence f is one-one.

- 2) $F: \mathbb{N} \cup \{-1, 0\} \rightarrow \mathbb{N}$, $f(n) = n + 2$

Answer: The function is one-one. If m is in co domain, then $m - 2$ is in domain and $f(m - 2) = (m - 2) + 2 = m$; thus m has a pre-image. Therefore, the function is onto. Also, it is bijective

- 3) $F: \mathbb{N} \rightarrow \mathbb{N}$, $f(n) = n^2$.

Answer: $f(m) = f(n) \Rightarrow m^2 = n^2 \Rightarrow m = n$. since $m, n \in \mathbb{N}$.

- 4) $F: \mathbb{R} \rightarrow \mathbb{R}$, $f(n) = n^2$.

Answer: $(1, 1), (-1, 1)$ etc. Hence two different elements of domain has the same image. It is not a function.

- 5) $F(x) = 1/x$, $f: \mathbb{R} \rightarrow \mathbb{R}$

Answer: $f(x)$ is not defined when $x = 0$, hence it is not a function from \mathbb{R} to \mathbb{R} .

- 6) What is the domain of f , when $f(x) = 1/x$, so that it is a function; is f onto?

Answer: since $1/x$ is undefined at $x = 0$, define domains as $\mathbb{R} - \{0\}$ with range $= \mathbb{R}$, then f is a function and is not onto as 0 in co domain has no preimage in $\mathbb{R} - \{0\}$. Hence f is not onto.

- 7) If $f: [-2, 2] \rightarrow \mathbb{B}$ given that $f(x) = 2x^3$. Find \mathbb{B} so that f is onto?

Answer: $f(-2) = -16$ and $f(2) = 16$. So $-16 \leq f(x) \leq 16$. Hence $\mathbb{B} = [-16, 16]$.

- 8) $F(x) = x|x|$, $x \in [-2, 2]$, is f one-one? If so find suitable co-domain so that function become *bijective*.

Answer: Let $x, y \in [-2, 2]$ such that $f(x) = f(y)$.

When $y = 0, x = 0$.

Let $x, y \neq 0, f(x) = f(y) \Rightarrow x|x| = y|y|$

$$\frac{x}{y} = \left| \frac{x}{y} \right|, \text{ since } \frac{x}{y} > 0, \text{ thus both } x \text{ and } y \text{ are negative or positive, hence } x^2 = y^2.$$

This is possible only if $x = y$. Thus f is one-one. When $x < 0$ $f(x) = -x^2$ and when $x \geq 0$, $f(x) = x^2$.

So the range is $[-4, 4]$. So f is bijective as range = co -domain, co-domain = $[-4, 4]$.

- 9) Find the largest possible domain for the real valued function f defined by $f(x) = \sqrt{x^2 - 5x + 6}$.

Answer for $\sqrt{f(x)}, f(x) > 0$, hence $x^2 - 5x + 6 \geq 0$. Factorizing this quadratic we get $(x - 2)(x - 3) \geq 0$

Using number line $-\infty \longleftarrow \longrightarrow \infty$, we get $(-\infty, 2), (2, 3), (3, \infty)$. $F(x)$ is positive in $(-\infty, 2] \cup [3, \infty) =$ domain.

10) Find the domain of (a) $\frac{1}{1-2\cos x}$ (b) range of $\frac{1}{1-3\cos x}$.

Answer: (a) function $\frac{1}{1-2\cos x}$ is defined except $1-2\cos x = 0$. So $\cos x \neq \frac{1}{2}$. Domain is $\mathbb{R} - \{2n\pi \pm \pi/3\}$, $n \in \mathbb{Z}$.

(b) finding range is done as:

$$-1 \leq \cos x \leq 1.$$

multiply by -3 we get $3 \geq -3\cos x \geq -3$. or $-3 \leq -3\cos x \leq 3$

$1-3 \leq 1-3\cos x \leq 1+3 \Rightarrow -2 \leq 1-3\cos x \leq 4$. Reciprocate we get $\frac{1}{1-3\cos x} \leq -1/2$ and $\frac{1}{1-3\cos x} \geq 1/4$.

Hence the range is $(-\infty, -1/2) \cup (1/4, \infty)$.

Long Answers:

1. Prove that function $f: \mathbb{N} \rightarrow \mathbb{N}$, defined by $f(x) = x^2 + x + 1$ is one-one but not onto. Find inverse of $f: \mathbb{N} \rightarrow S$, where S is range of f .

Solution:

Let $x_1, x_2 \in \mathbb{N}$.

Now, $f(x_1) = f(x_2)$

$$\begin{aligned} \Rightarrow x_1^2 + x_1 + 1 &= x_2^2 + x_2 + 1 \\ \Rightarrow x_1^2 + x_1 &= x_2^2 + x_2 \\ \Rightarrow (x_1^2 - x_2^2) + (x_1 - x_2) &= 0 \\ \Rightarrow (x_1 - x_2) \cdot (x_1 + x_2 + 1) &= 0 \\ \Rightarrow x_1 - x_2 = 0 & \quad [\because x_1 + x_2 + 1 \neq 0] \\ \Rightarrow x_1 &= x_2. \end{aligned}$$

Thus, f is one-one.

Let $y \in \mathbb{N}$, then for any x ,

$$f(x) = y \text{ if } y = x^2 + x + 1$$

$$\Rightarrow y = \left(x^2 + x + \frac{1}{4}\right) + \frac{3}{4}$$

$$\Rightarrow y = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$$

$$\Rightarrow x + \frac{1}{2} = \pm \sqrt{y - \frac{3}{4}}$$

$$\Rightarrow x = \pm \frac{\sqrt{4y-3}}{2} - \frac{1}{2}$$

$$\Rightarrow x = \frac{\pm\sqrt{4y-3}-1}{2}$$

$$\Rightarrow x = \frac{\sqrt{4y-3}-1}{2}$$

$$\left[\frac{-\sqrt{4y-3}-1}{2} \notin N \text{ for any value of } y \right]$$

Now, for $y = 3/4$, $x = -1/2 \notin N$.

Thus, f is not onto.

$\Rightarrow f(x)$ is not invertible.

Since, $x > 0$, therefore, $\frac{\sqrt{4y-3}-1}{2} > 0$

$$\Rightarrow \sqrt{4y-3} > 1$$

$$\Rightarrow 4y - 3 > 1$$

$$\Rightarrow 4y > 4$$

$$\Rightarrow y > 1.$$

Redefining, $f : (0, \infty) \rightarrow (1, \infty)$ makes

$f(x) = x^2 + x + 1$ on onto function.

Thus, $f(x)$ is bijection, hence f is invertible and $f^{-1} : (1, \infty) \rightarrow (0, \infty)$

$$f^{-1}(y) = \frac{\sqrt{4y-3}-1}{2}.$$

2. Prove that the function $f: [0, \infty) \rightarrow \mathbb{R}$ given by $f(x) = 9x^2 + 6x - 5$ is not invertible. Modify the co-domain of the function f to make it invertible, and hence find f^{-1} . (C.B.S.E. Sample Paper 2018-19)

Solution:

3. Let $y \in \mathbb{R}$.

For any x , $f(x) = y$ if $y = 9x^2 + 6x - 5$

$$\Rightarrow y = (9x^2 + 6x + 1) - 6$$

$$= (3x + 1)^2 - 6$$

$$\Rightarrow 3x + 1 = \pm\sqrt{y+6}$$

$$\Rightarrow x = \frac{\pm\sqrt{y+6}-1}{3}$$

$$\Rightarrow x = \frac{\sqrt{y+6}-1}{3}$$

$$\left[\because \frac{-\sqrt{y+6}-1}{3} \notin [0, \infty) \text{ for any value of } y \right]$$

For $y = -6 \in \mathbb{R}$, $x = -13 \notin [0, \infty)$.

Thus, $f(x)$ is not onto.

Hence, $f(x)$ is not invertible.

$$\text{Since, } x \geq 0, \therefore \frac{\sqrt{y+6}-1}{3} \geq 0$$

$$\Rightarrow \sqrt{y+6}-1 \geq 0$$

$$\Rightarrow \sqrt{y+6} \geq 1$$

$$\Rightarrow y+6 \geq 1$$

$$\Rightarrow y \geq -5.$$

We redefine,

$$f: [0, \infty) \rightarrow [-5, \infty),$$

which makes $f(x) = 9x^2 + 6x - 5$ an onto function.

Now, $x_1, x_2 \in [0, \infty)$ such that $f(x_1) = f(x_2)$

$$\Rightarrow (3x_1 + 1)^2 = (3x_2 + 1)^2$$

$$\Rightarrow [(3x_1 + 1) + (3x_2 + 1)][(3x_1 + 1) - (3x_2 + 1)]$$

$$\Rightarrow [3(x_1 + x_2) + 2][3(x_1 - x_2)] = 0$$

$$\Rightarrow x_1 = x_2$$

$$[\because 3(x_1 + x_2) + 2 > 0]$$

Thus, $f(x)$ is one-one.

$\therefore f(x)$ is bijective, hence f is invertible

and $f^{-1}: [-5, \infty) \rightarrow [0, \infty)$

$$f^{-1}(y) = \frac{\sqrt{y+6}-1}{3}.$$

$$\text{Also } f^{-1}(y) = \frac{\sqrt{y+6}-1}{3}$$

$$\Rightarrow f^{-1}(x) = \frac{\sqrt{x+6}-1}{3}$$

$$\therefore f^{-1}(43) = \frac{\sqrt{49}-1}{3} = \frac{7-1}{3} = 2$$

$$\text{and } f^{-1}(163) = \frac{\sqrt{169}-1}{3} = \frac{13-1}{3} = 4.$$

4. Let $f: A \rightarrow B$ be a function defined as $f(x) = 2x+3x-3$ where $A = \mathbb{R} - \{3\}$ and $B = \mathbb{R} - \{2\}$.

Is the function 'f' one-one and onto ?

Is 'f' invertible? If yes, then find its inverse.

Solution:

Let $x_1, x_2 \in A = \mathbb{R} - \{3\}$.

Now, $f(x_1) = f(x_2)$

$$\Rightarrow 2x_1+3x_1-3=2x_2+3x_2-3$$

$$\Rightarrow (2x_1 + 3)(x_2 - 3) = (2x_2 + 3)(x_1 - 3)$$

$$\Rightarrow 2x_1x_2 - 6x_1 + 3x_2 - 9 = 2x_1x_2 - 6x_2 + 3x_1 - 9$$

$$\Rightarrow -6x_1 + 3x_2 = -6x_2 + 3x_1$$

$$\Rightarrow 9x_1 = 9x_2$$

$$\Rightarrow x_1 = x_2$$

Thus, 'f' is one-one.

Let $y \in \mathbb{R} - \{2\}$.

Let $y = f(x_0)$.

$$\text{Then } 2x_0+3x_0-3 = y$$

$$\Rightarrow 2x_0 + 3 = x_0y - 3y$$

$$\Rightarrow x_0(y - 2) = 3(y + 1)$$

$$\Rightarrow x_0 = 3(y+1)y-2$$

Now, $y \in \mathbb{R} - \{2\} \Rightarrow 2x_0 + 3x_0 - 3 \in \mathbb{R} - \{2\}$

$$\begin{aligned} \therefore f(x_0) &= \frac{2x_0 + 3}{x_0 - 3} \\ &= \frac{2 \frac{3(y+1)}{y-2} + 3}{3 \frac{(y+1)}{y-2} - 3} \\ &= \frac{6y + 6 + 3y - 6}{3y + 3 - 3y + 6} = \frac{9y}{9} = y. \end{aligned}$$

5. Thus, 'f' is onto.

Hence, 'f' is one-one onto and consequently 'f' is invertible.

$$\text{Also } y = 2x + 3x - 3$$

$$xy - 3y = 2x + 3$$

$$x(y - 2) = 3(y + 1)$$

$$x = 3(y+1)/(y-2)$$

$$f^{-1}(y) = 3(y+1)/(y-2)$$

Hence, $f^{-1}(x) = 3(x+1)/(x-2)$ for all $x \in \mathbb{R} - \{2\}$.

6. Let $A = \mathbb{R} - \{2\}$ and $B = \mathbb{R} - \{1\}$. If $f: A \rightarrow B$ is a function defined by :

$f(x) = x - 1/x - 2$ Show that f is one one and onto.

Hence, find f^{-1} .

Solution:

(i) One-one : Let $x_1, x_2 \in \mathbb{R} - \{2\}$ such that

$$f(x_1) = f(x_2)$$

$$\Rightarrow x_1 - 1/x_1 - 2 = x_2 - 1/x_2 - 2$$

$$\Rightarrow x_1 x_2 - 2x_1 - x_2 + 2 = x_1 x_2 - 2x_2 - x_1 + 2$$

$$\Rightarrow x_1 = x_2$$

Thus one-one.

Onto : Let $f(x) = y$.

$$\text{Thus } x - 1/x - 2 = y \Rightarrow x = (2y - 1)/(y - 1)$$

\therefore Range of $f = \mathbb{R} - \{1\}$

= Co-domain of B

Thus f is onto.

$$(ii) f^{-1}(y) = 2y - 1/y - 1$$

$$\text{Hence } f^{-1}(x) = (2x - 1)/(x - 1)$$

7. Let $f : W \rightarrow W$ be defined by:

$$f(n) = \begin{cases} n - 1, & \text{if } n \text{ is odd} \\ n + 1, & \text{if } n \text{ is even} \end{cases}$$

Show that 'f' is invertible. Find the inverse of 'f'. (Here 'W' is the set of whole numbers)
(A.I.C.B.S.E. 2015)

8. Solution:

We have : $f : W \rightarrow W$ defined by :

$$f(n) = \begin{cases} n - 1, & \text{if } n \text{ is odd} \\ n + 1, & \text{if } n \text{ is even} \end{cases}$$

f is one-one.

When n_1 and n_2 are both odd,

$$\text{then } f(n_1) = f(n_2)$$

$$\Rightarrow n_1 - 1 = n_2 - 1$$

$$\Rightarrow n_1 = n_2$$

When n_1 and n_2 are both even, then

$$f(n_1) = f(n_2)$$

$$\Rightarrow n_1 + 1 = n_2 + 1$$

$$\Rightarrow n_1 = n_2$$

\therefore In both cases,

$$f(n_1) = f(n_2)$$

$$\Rightarrow n_1 = n_2$$

When n_1 is odd and n_2 is even,

then $f(n_1) = n_1 - 1$, which is even

and $f(n_2) = n_2 + 1$, which is odd.

$$\therefore (n_1) \neq (n_2)$$

$$\Rightarrow f(n_1) \neq f(n_2)$$

Similarly when n_1 is even and n_2 is odd,

$$\therefore (n_1) \neq (n_2)$$

$$\Rightarrow f(n_1) \neq f(n_2)$$

In each case, 'f' is one-one.

f is onto.

When n is odd whole number, then there exists an even whole number

$n - 1 \in W$ such that

$$f(n - 1) = (n - 1) + 1 = n.$$

When n is even whole number, then there exists an odd whole number $n + 1 \in W$ such that

$$f(n + 1) = (n + 1) - 1 = n.$$

Also $f(1) = 0 \in W$.

\therefore each number of W has its pre-image in W .

Thus f is onto.

Hence, ' f ' is one-one onto

\Rightarrow ' f ' is invertible.

To obtain f^{-1} .

Let $n_1, n_2 \in W$

such that $f(n_1) = f(n_2)$

$n_1 + 1 = n_2$, if n_1 is even

$n_1 - 1 = n_2$, if n_1 is odd.

$$\text{Thus } n_1 = \begin{cases} n_2 - 1, & \text{if } n_2 \text{ is odd} \\ n_2 + 1, & \text{if } n_2 \text{ is even} \end{cases}$$

$$\Rightarrow f^{-1}(n_2) = \begin{cases} n_2 - 1, & \text{if } n_2 \text{ is odd} \\ n_2 + 1, & \text{if } n_2 \text{ is even} \end{cases}$$

$$\text{Then } f^{-1}(n_1) = \begin{cases} n_1 + 1, & \text{if } n_1 \text{ is even} \\ n_1 - 1, & \text{if } n_1 \text{ is odd} \end{cases}$$

Hence $f = f^{-1}$

Chapter -2

Trigonometric Functions & Inverse Trigonometric functions:

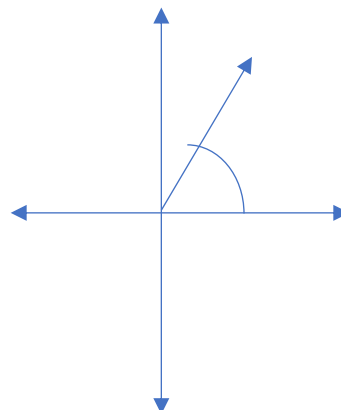
Study of angle measures, angles of a triangles, ratio of sides of a right angled triangle is *Trigonometry*.

Angles are measured in degree, radians.

Using cartesian co ordinate systems angles classified into :

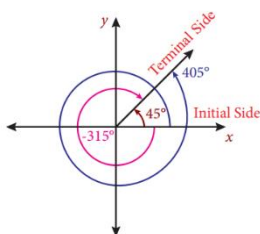
Angles measured in anti-clockwise directions are taken positive angles, angles measured in clockwise directions are negative angles.

Quadrant	Angles	Intervals θ is measure of angle
1 st	0° to 90°	$0^\circ \leq \theta \leq 90^\circ$
2 nd	90° to 180°	$90^\circ < \theta \leq 180^\circ$
3 rd	180° to 270°	$180^\circ < \theta \leq 270^\circ$
4 th	270° to 360°	$270^\circ < \theta \leq 360^\circ$
	$n360 \pm \theta$, n is an Integer	



Note: Vertex is at origin, terminal sides fall in any quadrant.

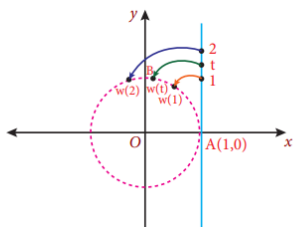
Angles in standard position having terminal sides along x -axis or y -axis are called quadrantal angles. $0, 90, 180, 270, 360$ are quadrantal angles.



about co-terminal angles.

Definition of **basic Trigonometric ratios using co-ordinate system as a function:**

We have learnt about six trigonometric ratios $\sin\theta, \cos\theta, \tan\theta, \operatorname{cosec}\theta, \sec\theta$ and $\cot\theta$.



Consider (x, y) such that $y = \sin x$ then this becomes a trigonometric function.

Reassociated with the point $A(1, 0)$ on the unit circle. Draw a tangent to the unit circle at the point $A(1, 0)$. Let t be a real number such that t is y -coordinate of a point on the tangent line

Any point $B(x, y)$ on the unit circle is $B(\cos\theta, \sin\theta)$. Here θ is measured in radian which is a real number. Hence domain and range of trigonometric functions are given below:

Trigonometric Function	Domain	Range
$F(x) = \sin x; y = \sin x$	$x \in \mathbb{R}$	$-1 \leq \sin x \leq 1; y \in [-1, 1]$
$F(x) = \cos x; y = \cos x$	$x \in \mathbb{R}$	$-1 \leq \cos x \leq 1; y \in [-1, 1]$
$F(x) = \tan x; y = \tan x$	$x \in \mathbb{R}$	$-\infty < \tan x < \infty; y \in \mathbb{R}$
$F(x) = \operatorname{Cosec} x$	$x \in \left[\frac{-\pi}{2}, \frac{\pi}{2} \right] - \{0\}$	$\mathbb{R} - (-1, 1)$

A function has inverse if and only if it is one and onto. All the trigonometric functions are many one. However, if we restrict the domain suitably, we can make the function *one-one* in the restricted domain. The following is the restricted domain for inverse trigonometric functions.

Notation: *inverse trigonometric functions* is denoted by $f(x) = \sin^{-1}x$ or $\arcsin x$ etc.,

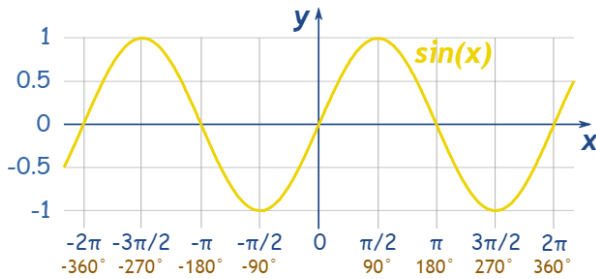
Trigonometric function	Inverse trigonometric function
$\sin x : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \rightarrow [-1, 1]$	$\sin^{-1} x : [-1, 1] \rightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos x : [0, \pi] \rightarrow [-1, 1]$	$\cos^{-1} x : [-1, 1] \rightarrow [0, \pi]$
$\tan x : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \rightarrow (-\infty, \infty)$	$\tan^{-1} x : (-\infty, \infty) \rightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$\operatorname{Cosec} x : \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\} \rightarrow \mathbb{R} - (-1, 1)$	$\operatorname{Cosec}^{-1} x : \mathbb{R} - (-1, 1) \rightarrow \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
$\sec x : [0, \pi] - \left\{\frac{\pi}{2}\right\} \rightarrow \mathbb{R} - (-1, 1)$	$\sec^{-1} x : \mathbb{R} - (-1, 1) \rightarrow [0, \pi] - \left\{\frac{\pi}{2}\right\}$
$\cot x : (0, \pi) \rightarrow (-\infty, \infty)$	$\cot^{-1} x : (-\infty, \infty) \rightarrow (0, \pi)$

Principal values of inverse trigonometric functions are listed below:

Principal value for $x \geq 0$	Principal value for $x < 0$
$0 \leq \sin^{-1}(x) \leq \frac{\pi}{2}$	$-\frac{\pi}{2} \leq \sin^{-1}(x) < 0$
$0 \leq \cos^{-1}(x) \leq \frac{\pi}{2}$	$\frac{\pi}{2} < \cos^{-1}(x) \leq \pi$
$0 \leq \tan^{-1}(x) < \frac{\pi}{2}$	$-\frac{\pi}{2} < \tan^{-1}(x) < 0$
$0 < \cot^{-1}(x) \leq \frac{\pi}{2}$	$-\frac{\pi}{2} < \cot^{-1}(x) < 0$
$0 \leq \sec^{-1}(x) < \frac{\pi}{2}$	$\frac{\pi}{2} < \sec^{-1}(x) \leq \pi$
$0 < \operatorname{cosec}^{-1}(x) \leq \frac{\pi}{2}$	$-\frac{\pi}{2} < \operatorname{cosec}^{-1}(x) < 0$

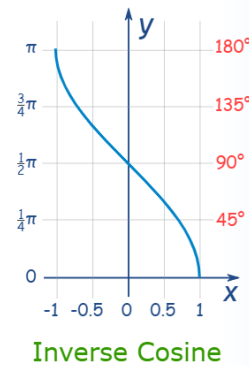
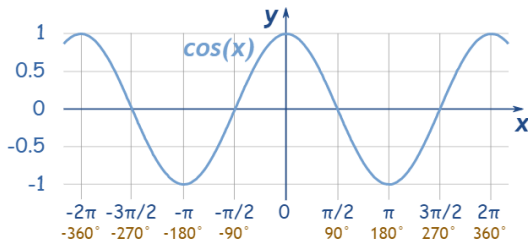
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Graph of trigonometric functions:



It can be verified that $\sin^{-1}x$ graph is reflection about the line $y = x$.

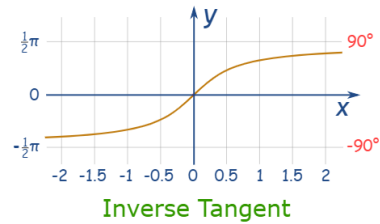
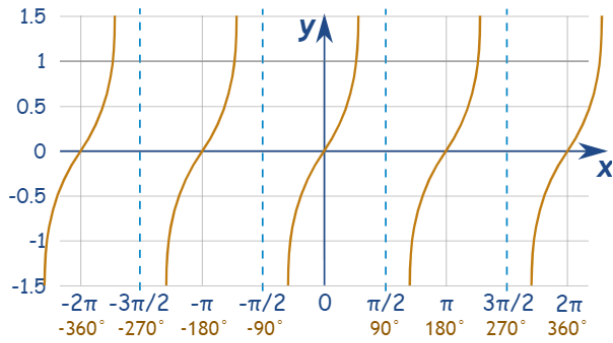
Graph of cosine and inverse of cosine:



Graph of Tangent and inverse tangent functions:

The Tangent function has a completely different shape ... it goes between negative and positive Infinity, crossing through 0, and at every π radians (180°), as shown on this plot.

At $\pi/2$ radians (90°), and at $-\pi/2$ (-90°), $3\pi/2$ (270°), etc, the function is officially **undefined**, because it could be **positive Infinity** or **negative Infinity**.



Properties of inverse trigonometry functions:

1. $\sin(\sin^{-1}x) = x, x \in [-1, 1]$
2. $\sin^{-1}(\sin x) = x, x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$
3. $\cos(\cos^{-1}x) = x, x \in [-1, 1]$
4. $\cos^{-1}(\cos x) = x, x \in [0, \pi]$
5. $\tan(\tan^{-1}x) = x, x \in \mathbb{R}$
6. $\tan^{-1}(\tan x) = x, x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$
7. $\operatorname{cosec}(\operatorname{cosec}^{-1}x) = x$
8. $\operatorname{cosec}^{-1}(\operatorname{cosec} x) = x$
9. $\sec(\sec^{-1}x) = x$
10. $\sec^{-1}(\sec x) = x$
11. $\cot(\cot^{-1}x) = x$
12. $\cot^{-1}(\cot x) = x$
13. $\sin^{-1}x + \cos^{-1}x = \pi/2, x \in [-1, 1]$
14. $\tan^{-1}x + \cot^{-1}x = \pi/2, x \in \mathbb{R}$
15. $\operatorname{cosec}^{-1}x + \sec^{-1}x = \pi/2, |x| \geq 1$

Problems:

1. Find the principal value of (i) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ (ii) $\operatorname{cosec}^{-1}(2/\sqrt{3})$

Answer:

Let $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = y$ where $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \Rightarrow \sin y = \left(\frac{\sqrt{3}}{2}\right) \Rightarrow y = 60^\circ = \pi/3$.

Let $\operatorname{cosec}^{-1}(2/\sqrt{3}) = y$, then $\operatorname{cosec} y = 2/\sqrt{3} \Rightarrow \sin y = \sqrt{3}/2$. Answer $y = \pi/3$.

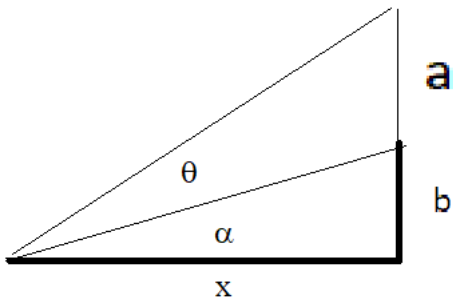
2. Find the principal value of $\tan^{-1}\frac{-1}{\sqrt{3}}$?

Answer: $\tan y = -1/\sqrt{3} \Rightarrow \tan(-60^\circ)$, solution: $y = -60^\circ$ or $-\pi/6$.

3. A man standing directly opposite to one side of a road of x meter views a circular shaped traffic signal of diameter a meter on the other side of the road. The bottom of the green signal of b meter height of viewers

eye level. If θ denotes the angle subtended by the diameter of the green signal, then prove that $\theta =$

$$\tan^{-1}\left(\frac{a+b}{x}\right) - \tan^{-1}\left(\frac{b}{x}\right).$$



Solution :

MCQ:

$$\tan(\theta+\alpha) = \frac{a+b}{x}$$

$$\tan \alpha = b/x$$

There fore

$$\theta = \tan^{-1}\left(\frac{a+b}{x}\right) - \tan^{-1}\left(\frac{b}{x}\right).$$

1. If $\tan^{-1}(\cot x) = 2x$, then x is (a) $\pi/3$ (b) $\pi/4$ (c) $\pi/6$ (d)
2. None of these
3. $\cot(\pi/4 - 2\cot^{-1}3) =$ (a) 7 (b) 6 (c) 5 (d) none of these
4. $\sin^{-1}(0.5) =$ (a) $\pi/3$ (b) $-\pi/3$ (c) $\pi/6$ (d) $-\pi/6$
5. $\cos^{-1}(0.5) =$ (a) $-\pi/3$ (b) $\pi/3$ (c) $\pi/2$ (d) $2\pi/3$
6. $\sin^{-1}(1/2) + \cos^{-1}(1/2) =$ (a) $\pi/2$ (b) $-\pi/2$ (c) π (d) $-\pi$
7. $\tan^{-1}(1) + \cos^{-1}(1/2) + \sin^{-1}(1/2) =$ (a) $2\pi/3$ (b) $3\pi/4$ (c) $\pi/2$ (d) 6π
8. If $\cot^{-1}(\sqrt{\cos \alpha}) + \tan^{-1}(\sqrt{\cos \alpha}) = x$, then $\sin x$ is (a) $\tan 2(\alpha/2)$ (b) $\cot 2(\alpha/2)$ (c) $\tan \alpha$ (d) $\cot(\alpha)$
9. The value of $\cot(\operatorname{cosec}^{-1}(5/3) + \tan^{-1}(2/3)) =$ (a) $5/4$ (b) $6/17$ (c) $3/17$ (d) $4/17$
10. If $6 \sin^{-1}(x^2 - 6x + 8) = \pi$, then x is (a) 1 (b) 2 (c) 6 (d) 8
11. $\sin(2 \cos^{-1}(3/5)) =$ (a) $6/25$ (b) $24/25$ (c) $4/5$ (d) $-24/25$
12. $\sin^{-1}(1-x) - 2 \sin^{-1}x = \pi/2$, $x =$ (a) 0 (b) $1/2$ (c) $-1/2$ (d) none of these
13. $\sin[\cot^{-1}(\cos(\tan^{-1}x))] =$ (a) $\sqrt{\frac{x^2-1}{x^2+2}}$ (b) $\sqrt{\frac{x^2-1}{x^2-2}}$ (c) $\sqrt{\frac{x-1}{x-2}}$ (d) $\sqrt{\frac{x+1}{x+2}}$
14. Prove that $\tan^{-1}\sqrt{x} = \frac{1}{2} \cos^{-1} \frac{1-x}{1+x}$
15. $\cot^{-1}\left(\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}\right) = \frac{x}{2}, x \in \left(0, \frac{\pi}{4}\right)$
16. $\tan^{-1}\left(\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}}\right) = \frac{\pi}{4} - \frac{1}{2} \cos^{-1}x, -\frac{1}{\sqrt{2}} \leq x \leq 1$ hint Put $x = \cos 2\theta$
17. solve $2 \tan^{-1}(\cos x) = \tan^{-1}(2 \operatorname{cosec} x)$

Answers:

MCQ:

1) C 2) a 3) d 4) b 5) a 6) b 7) a 8) b 9) b 10) d 11) a 12) a 13) a

14) Let $\tan^{-1}\sqrt{x} = \theta$

$$\text{therefore } \tan\theta = \sqrt{x} \Rightarrow \cos\theta = \frac{1}{\sqrt{1+x}}$$

$$\text{Now } \cos 2\theta = 2\cos^2\theta - 1 \Rightarrow 2\left(\frac{1}{\sqrt{1+x}}\right)^2 - 1 = \frac{1-x}{1+x}$$

$$\text{Therefore } 2\theta = \cos^{-1}\frac{1-x}{1+x}$$

$$\text{Hence } \tan^{-1}\sqrt{x} = \frac{1}{2} \cos^{-1}\frac{1-x}{1+x}$$

15) we use $1 \pm 2 \sin x = (\sin^2\frac{x}{2} + \cos^2\frac{x}{2}) \pm 2 \sin\frac{x}{2} \cos\frac{x}{2} \Rightarrow (\sin\frac{x}{2} \pm \cos\frac{x}{2})^2$

$$\text{Hence } \cot^{-1}\left(\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}\right) = \cot^{-1}\left(\frac{(\sin\frac{x}{2} + \cos\frac{x}{2}) + (\sin\frac{x}{2} - \cos\frac{x}{2})}{(\sin\frac{x}{2} + \cos\frac{x}{2}) - (\sin\frac{x}{2} - \cos\frac{x}{2})}\right) = \cot^{-1}\left(\tan\frac{x}{2}\right) = \cot^{-1}\left(\cot\left(\frac{\pi}{2} - \frac{x}{2}\right)\right) =$$

$$\frac{\pi}{2} - \frac{x}{2}$$

Hence answer.

17) Let $\tan^{-1}x = \theta \Rightarrow \tan\theta = x$

$$\text{Now } \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

$$\text{Replacing } \tan\theta \text{ with } x, \text{ we get } \tan 2\theta = \frac{2x}{1-x^2}$$

$$\text{Therefore } 2\theta = \tan^{-1}\frac{2x}{1-x^2}$$

$$\text{Now } 2\tan^{-1}(\cos x) = \tan^{-1}\frac{2\cos x}{1-\cos^2 x} = \tan^{-1}\frac{2\cos x}{\sin^2 x}$$

$$\text{Therefore } \frac{2\cos x}{\sin^2 x} = 2\operatorname{cosec} x \Rightarrow \cos x = \frac{\sin^2 x}{\sin x} \Rightarrow \tan x = 1, x = 45^\circ \text{ or } \pi/4.$$

Chapter 3: Matrices:

Definition: A **matrix** is an ordered rectangular array of numbers or functions.

Elements of a Matrix: the numbers or functions in a matrix are called elements.

How is a matrix named? The matrix is named using capital letters.

What is order of a matrix?

A matrix having m rows and n columns is called a of order $m \times n$ (read as m by n).

Number of elements of a matrix of order $m \times n = mn$. We write matrix as $\mathbf{A}_{m \times n}$.

Every element of a matrix is denoted by a_{ij} where i represents rows, j represents columns.

Example: a_{23} means element at the cross section of 2nd row and 3rd column.

Example s:

$A_{2 \times 3} = \begin{bmatrix} 5 & 4 & 2 \\ 0 & 1 & 3 \end{bmatrix}$, here 2nd row, 3rd column element is 3. 0 belongs to a_{21} .

Row Matrix: Matrix having 1 row and many columns is a row matrix.

$$A_{1 \times 3} = [1 \quad 3 \quad 2]_{1 \times 3}$$

Column Matrix: Matrix having 1 column and many rows is a column matrix:

$$A_{3 \times 1} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}_{3 \times 1}$$

What does matrix represent?

Each element of a matrix represent a value assigned as shown:

Consider the following:

Number of men and women workers in two factories I, II and III as in the table:

	Men	Women
I	30	25
II	25	31
III	27	26

This table is represented as a 3×2 matrix

$$W_{3 \times 2} = \begin{bmatrix} 30 & 25 \\ 25 & 31 \\ 27 & 26 \end{bmatrix}_{3 \times 2}$$

Types of matrices:

1. Row matrix
2. Column Matrix.
3. Square Matrix, matrix having same number of rows and columns . $A_{2 \times 2}$, $B_{3 \times 3}$ etc.,

4. Diagonal Matrix: All of its non-diagonal elements are **zeros**. $A_{3 \times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

5. Scalar Matrix: In a diagonal matrix if the diagonal elements are equal(same). $B_{3 \times 3} = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} \end{bmatrix}$

6. Identity Matrix: In a diagonal matrix if diagonal elements are 1 (unity) , then it s an identity matrix.

$$C_{3 \times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \text{ Every identity matrix is a scalar matrix.}$$

7. Zero Matrix: If all the elements of a matrix are **zeros**, it is called **zero** matrix.
8. Triangular or upper or lower triangular matrix: A special type of square matrix where all the values above or below the main diagonal are zero. $L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & 0 \\ 4 & 5 & 6 \end{bmatrix}$ is called lower triangular matrix, $U = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ is called upper triangular matrix. *Usage: To solve simultaneous equations by back substitution method.*
9. Multiplication of a matrix by a scalar: Let $A = [a_{ij}]$ then $kA = [ka_{ij}]$.
10. Negative matrix: In (9) if $k = -1$ then each element of multiplied by -1 .
11. Equal matrices: Let A and B be two matrices of same **order**, they are equal if element wise are equal.

Example $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2 \times 2}$ and $B = \begin{bmatrix} 1 & +\sqrt{4} \\ 18/6 & 2^2 \end{bmatrix}$, we find elementwise they are equal, hence Matrix $A =$ matrix B .

Operations on Matrices:

Adding or subtracting two matrices :

- (i) Let A, B be matrix of same order such that $A = [a_{ij}]_{3 \times 2}$, $B = [b_{ij}]_{3 \times 2}$, the $C_{3 \times 2} = A + B = [a_{ij} + b_{ij}]_{3 \times 2}$.
- (ii) $C_{3 \times 2} = A - B = [a_{ij} - b_{ij}]_{3 \times 2}$.

Properties of addition of two Matrices: (problems numerical are give in exercise for proving!)

- (i) **Addition of two matrices is commutative: $A + B = B + A$**
- (ii) **Addition is associative Matrices A, B and C of same order obeys $(A+B)+C = A+(B+C)$**
- (iii) **Additive identity: For every matrix A , the zero matrix of same order obeys $A+O=O+A$; O is additive identity of matrix A**

Properties of scalar multiplication:

- (i) **$K(A+B) = kA + kB$**
- (ii) **$(k + I) A = kA + IA$**

Multiplication of two matrices:

- (1) **Multiplying two or more matrices must obey the following**

Compatibility of multiplication: *For multiplying matrix A with matrix B they must have*
:Number of columns of $A =$ number of rows of B .

- (2) **Let A be order $m \times n$ and B be order $n \times p$ then the product $AB = C$ will be order $m \times p$.**

- (3) **If $A = [a_{ij}]$, $B = [b_{jk}]$ then $c_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + a_{i3}b_{3k} + \dots + a_{in}b_{nk} = \sum_{j=1}^n a_{ij}b_{jk}$.**

Properties of Multiplication of Matrices:

- i) $AB \neq BA$ (*non commutative*)
- ii) Two non zero matrices, when multiplied can become zero.
- iii) IF A, B, C are such they obey *compatibility of multiplication* then $(AB)C = A(BC)$. This property is called *associative property of multiplication*.
- iv) The distributive property of (1) Multiplication over addition: $A(B + C) = AB + AC$ (2) addition over multiplication: $(A + B)C = AC + BC$
- v) The identity matrix is called **Multiplicative identity**. Let I be the identity matrix and let A be a square matrix the $IA = AI = A$
- vi) $A^2 = A \times A$; $A^3 = A^2 \times A$ or $A \times A^2$.
- vii) $A^2 + mA + nI = O$, is a quadratic in A.

Transpose of Matrix:

If $A = [a_{ij}]$ of order $m \times n$, the matrix obtained by interchanging the rows and columns of A and denoted by $A' = [a_{ji}]_{n \times m}$ is called the **transpose** of A. Also transpose of A is denoted by A' or A^T

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \end{bmatrix} \text{ then } A' = \begin{bmatrix} 1 & 6 \\ 2 & 5 \\ 3 & 4 \end{bmatrix}.$$

Properties:

1. Transpose of a transpose of $A = A$; $(A')' = A$
2. $(kA)' = kA'$
3. $(A + B)' = A' + B'$
4. $(AB)^t = B^t \times A^t$
5. If a square matrix $A = A'$, then A is called symmetric matrix.
6. If a square matrix $A' = -A$, then A is called *skew* symmetric matrix.
 - a. All the diagonal elements of a *skew* symmetric matrix are zero.

Theorem -1: For any matrix A with *real number* elements, the $A + A'$ is symmetric and $A - A'$ is *skew*.

Theorem 2: $A = \frac{1}{2} (A + A') + \frac{1}{2} (A - A')$. Any square matrix can be expressed as sum of a symmetric and skew symmetric matrices.

We know that for a square matrix A, $AI = IA$ then there exists a matrix B such that $AB = BA = I$. This matrix B is called **inverse of A**.

How to find inverse of a square Matrix?

There two methods: (1) By elementary row/column transformation of matrix (*deleted portion*) (2) using the determinant and adjoint of matrix.

(vi) Orthogonal Matrix: A matrix A is said to be orthogonal if $AA^T = A^T A = I$

(vii) Nilpotent Matrix: $A = [a_{ij}]$ be an $n \times n$ matrix with $a_{ij} = \begin{cases} 1, & \text{if } i = j + 1 \\ 0, & \text{otherwise} \end{cases}$, then $A^n = O$ and $A^t \neq O$ for

$1 \leq t \leq n - 1$. The matrices A for which a positive integer k such that $A^k = O$ are called NILPOTENT matrices. The least value of k is called order of NILPOTENCY.

(viii) The matrices that satisfy $A^2 = A$ is called IDEMPOTENT matrices. Example: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

MCQ:

1. $A = [a_{ij}]_{m \times n}$ is a square matrix, if (a) $m < n$ (b) $m > n$ (c) $m = n$ (d) none
2. Which of the given values of x and y make the matrices equal $\begin{bmatrix} 3x + 1 & 5 \\ y + 1 & 2 - 3x \end{bmatrix} = \begin{bmatrix} 0 & y - 2 \\ 8 & 4 \end{bmatrix}$ (a) $x = -1/3, y = 7$ (b) not possible to find (c) $y = 7, x = -2/3$ (d) $x = -1/3, y = -2/3$
3. The number of possible matrices of order 3×3 with entries 0 or 1 is (a) 27 (b) 18 (c) 81 (d) 512
4. Assume X, Y, Z, W and P are matrices of order $2 \times n, 3 \times k, 2 \times p, n \times 3$ and $p \times k$ respectively chose the correct
 - a. The restriction on n, k and p so that $PY + WY$ will be defined are (a) $k = 3, p = n$ (b) k is arbitrary, $p = 2$ (c) p is arbitrary, $k = 3$ (d) $k = 2, p = 3$
 - b. If $n = p$, then the order of the matrix $7X - 5Z$ is (a) $p \times 2$ (b) $2 \times n$ (c) $n \times 3$ (d) $p \times n$
 5. If A, B are symmetric matrices of same order then $AB - BA$ is a (a) Skew symmetric matrix (b) symmetric matrix (c) Zero matrix (d) Identity matrix
6. If $A = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$, and $A + A' = I$, then the value of α is (a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) $\frac{3\pi}{2}$
7. Matrices A and B will be inverse of each other only if (a) $AB = BA$ (b) $AB - BA = 0$ (c) $AB = 0, BA = 1$ (d) $AB = BA = I$
8. If A is a matrix of order $m \times n$ and B is matrix such that AB^t and $B^t A$ are both defined then the order of B is (a) $m \times m$ (b) $n \times n$ (c) $n \times m$ (d) $m \times n$
9. The matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ is a (a) Identity matrix (b) symmetric matrix (c) skew symmetric (d) None of these.

10. The matrix $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 5 & 4 \end{bmatrix}$ is a (a) symmetric matrix (b) skew symmetric matrix (c) upper triangular (d)

lower triangular

11. If the matrix A is both symmetric and skew symmetric, then (a) A is a diagonal matrix (b) A is zero matrix (c) A is a square matrix (d) None of these

12. If $A = \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}$ is such that $A^2=I$ then (a) $1+\alpha^2+\beta\gamma=0$ (b) $1-\alpha^2+\beta\gamma=0$ (c) $1-\alpha^2-\beta\gamma=0$ (d) $1+\alpha^2-\beta\gamma=0$

13. If A is a square matrix such that $A^2=A$, then $(I+A)^2-7A$ is equal to (a) A (b) I-A (c) I (d) 3A

Short Answer:

- If A is of order 2×3 and B is of order 3×2 , $A=[a_{ij}]$ such that $a_{ij} = 2i+3j$ and $B=[b_{ij}]$ such $b_{ij} = 3i - 2j$, find the following :
 - A+B
 - AB
 - BA
 - is $(AB)^t = B^t A^t$?
 - $(AB)^{-1} = B^{-1} A^{-1}$
- If A and B are symmetric matrices of the same order, then show that AB is symmetric if and only if A and B commute.
- Let $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}$ find a matrix D such that $CD - AB = O$.
- If A and B are symmetric matrices prove that $AB - BA$ is a skew symmetric matrix.
- Show that the matrix $B^t A B$ is a symmetric matrix or skew matrix according as A is symmetric or skew symmetric.

Long answer type:

- If $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ then prove that $A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$, $n \in \mathbb{N}$
- Let $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}$ find a matrix D such that $CD - AB = O$
- Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, show that $(aI + bA)^n = a^n I + n a^{n-1} b A$, where I is the identity matrix of order 2 and $n \in \mathbb{N}$
- If $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, prove that $A^n = \begin{bmatrix} 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \end{bmatrix}$, $n \in \mathbb{N}$.
- If $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, then prove that $A^n = \begin{bmatrix} 1 + 2n & -4n \\ n & 1 - 2n \end{bmatrix}$

6. Find the value of x , y and z if the matrix $A = \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}$ satisfy $A^t A = I$

7. Find the value of x if $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ x \end{bmatrix} = O$?

8. If $A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = 0$, hence find A^{-1} .

9. Find the matrix X so that $X \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{bmatrix}$

10. If A and B are invertible matrices of same order, then $(AB)^{-1} = B^{-1}A^{-1}$, prove using the property $AA^{-1} = A^{-1}A = I$. Hence verify the same using the formula : $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then $A^{-1} = \frac{1}{|A|} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$, given that

$$A = \begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix}, B = \begin{bmatrix} -1 & 4 \\ 4 & -1 \end{bmatrix}$$

11. For the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 2 & -3 \\ 2 & 1 & 3 \end{bmatrix}$, show that $A^3 - 6A^2 + 5A + 11I = O$. Hence find A^{-1} .

12. The following is a numerical problem: Convert into matrix equation.

- The sum of three numbers is 6. If we multiply third number by 3 and add second number to it, we get 11. By adding first and third number we get double of the second number.
- The cost of 4Kg of onion, 3 Kg of wheat and 2 Kg of rice is Rs.60. The cost of 2Kg of onion, 4 Kg of wheat and 6 Kg of rice is Rs.90. The cost of 6Kg of onion, 2 Kg of wheat and 3 Kg of rice is Rs.70.
- markets. Annual sales are indicated below table:

Market	Products		
I	10,000	2,000	18,000
II	6,000	20,000	8,000

Convert to matrix.

- If unit sale prices of x , y and z are Rs. 2.50, Rs. 1.50 and Rs. 1.00 respectively. Find the total revenue in each market with the help of matrix multiplication algebra.
 - If the unit costs of the above three commodities are Rs. 2.00, Rs. 1.00 and Rs. 50 paise. Find the gross profit.
13. A trust fund has Rs. 30,000 that must be invested in two different types of bonds. The first bond pays 5% interest per year, and the second bond pays 7% interest per year. Using matrix multiplication, determine how to divide Rs. 30,000 among the two type of bonds. If the trust fund must obtain an annual interest of (a) Rs. 1800 (b) Rs. 2000.

14. The bookshop of a particular school has 10 dozen chemistry books, 8 dozen physics books, 10 dozen Economics books. Their selling prices are Rs. 80, Rs.60 and Rs 40 each respectively. Find the total amount the bookshop will receive from selling all the books using matrix algebra.

15. Show that if $A = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$, (a) $(A+B)(A+B) \neq A^2 + 2AB + B^2$ (b) $(A+B)(A-B) \neq A^2 - B^2$.

16. If $A = \begin{bmatrix} 2 & -2\sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 2\sqrt{2} \\ -\sqrt{2} & 2 \end{bmatrix}$, Verify commutative law of multiplication?

17. Express $A = \begin{bmatrix} \tan\theta & 1 \\ -1 & \cot\theta \end{bmatrix}$ as sum of a symmetric and skew symmetric matrices.

18. Which of the following are skew symmetric, hence verify the properties of skew symmetric matrix. (a)

$$A = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix} \text{ (b) } B = \begin{bmatrix} 0 & -1 & -2 \\ -1 & 0 & -3 \\ -2 & -3 & 0 \end{bmatrix} \text{ (c) } \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

19. Let $A = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{bmatrix}$, prove that A is orthogonal?

20. Let A and B be skew symmetric matrices with $AB = BA$. Is the matrix AB symmetric or skew symmetric?

21. If $A = \begin{bmatrix} 0 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 0 \end{bmatrix}$ and I is a unit matrix of order 2, show that $I + A = (I - A) \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

Answers/Solutions:

MCQ:

1) (c) 2. a 3. d 4. (a) a 4.(b) b 5. C 6. B 7. A 8. D 9. B 10. D 11. B 11. B 12. C 13.

Short Answer: solution

1. ..
a. Addition cannot be done (orders of addend are not equal)

b. $\begin{bmatrix} 114 & 51 \\ 138 & 57 \end{bmatrix}$

c. $\begin{bmatrix} 33 & 48 & 63 \\ 34 & 52 & 70 \\ 70 & 106 & 142 \end{bmatrix}$

d. Yes

e. No

2. As $A = A^t$, $B = B^t$ and if AB is symmetric then $(AB)^t = AB$

$$(AB)^t = AB \Rightarrow B^t A^t = AB$$

$BA = AB$, hence A, B commutes.

3. $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}$

Let matrix D such that $CD - AB = O$ (1)

Assume $D = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, substituting in (1)

We get $2a+5c-3=0$, $3a+8c-43=0$, $2b+5d=0$, $3b+8d-22=0$.

Solving for a, b, c, d we get $D = \begin{bmatrix} -186 & -110 \\ 75 & 144 \end{bmatrix}$,

4. If A and B are symmetric matrices prove that $AB - BA$ is a skew symmetric matrix.

Answer: Given $A=A'$, $B=B'$;

$$\begin{aligned} (AB-BA)' &= (AB)' - (BA)' \\ &= B'A' - A'B' \\ &= BA - AB \\ &= -(AB-BA) \end{aligned}$$

Hence $AB - BA$ is skew symmetric

5. If A be symmetric i.e., $A'=A$ then

$$(B'AB)' = [B'(AB)]' = (AB)'(B')' = (B'A')B = B'A'B = B'AB$$

Hence $B'AB$ is symmetric

If A is skew symmetric i.e., $A' = -A$ then

$$(B'AB)' = [B'(AB)]' = (AB)'(B')' = (B'A')B = B'(-A)B = -(B'AB)$$

$\therefore B'AB$ is skew symmetric

Long Answer :

1. $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ then prove that $A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$, $n \in \mathbb{N}$

Solution:

$$\begin{aligned}
&= \begin{bmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \\
&= \begin{bmatrix} \cos \theta \cos k\theta - \sin \theta \sin k\theta & \cos \theta \sin k\theta + \sin \theta \cos k\theta \\ -\sin \theta \cos k\theta + \cos \theta \sin k\theta & -\sin \theta \sin k\theta + \cos \theta \cos k\theta \end{bmatrix} \\
&= \begin{bmatrix} \cos(k\theta + \theta) & \sin(k\theta + \theta) \\ -\sin(k\theta + \theta) & \cos(k\theta + \theta) \end{bmatrix} = \begin{bmatrix} \cos(k+1)\theta & \sin(k+1)\theta \\ -\sin(k+1)\theta & \cos(k+1)\theta \end{bmatrix}
\end{aligned}$$

Therefore, the result is true for $n = k + 1$.

Thus by principle of mathematical induction, we have

$$A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}, n \in \mathbb{N}$$

We shall prove the result by using principle of mathematical induction.

$$\text{We have } P(n) : \text{ If } A = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix},$$

$$\text{then } A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}, n \in \mathbb{N}$$

Let $n = 1$, then $P(1) = A^1$

$$= \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

Therefore, the result is true for $n = 1$.

Let the result be true for $n = k$. So

$$P(k) = A^k = \begin{bmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{bmatrix}$$

Now, we prove that the result holds for $n = k + 1$

$$\text{i.e. } P(k+1) = A^{k+1} = \begin{bmatrix} \cos(k+1)\theta & \sin(k+1)\theta \\ -\sin(k+1)\theta & \cos(k+1)\theta \end{bmatrix}$$

$$\text{Now, } P(k+1) = A^{k+1} = A^k \cdot A$$

2. Try

3. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, show that $(aI + bA)^n = a^n I + na^{n-1}bA$, where I is the identity matrix of order 2 and $n \in \mathbb{N}$

Answer:

$$\text{Let } P(n) = (aI + bA)^n = a^n I + na^{n-1}bA$$

Put $n=1$

$$\text{LHS : } \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} + \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$$

RHS: $aI + 1a^{1-1}bA = aI + bA$, hence LHS = RHS. For $n=1$ $P(1)$ is true

Therefore the result is true for $n = k$.

$$P(k) : (aI + bA)^k = a^k I + k a^{k-1} b A - 1 b A$$

Now we prove that the result is true for $n = k + 1$

Consider

$$\begin{aligned} (aI + bA)^{k+1} &= (aI + bA)^k (aI + bA) \\ &= (a^k I + k a^{k-1} b A)(aI + bA) \\ &= a^{k+1} I + k a^k b A I + a^k b I A + k a^k b A^2 + b^2 A^2 \\ &= a^{k+1} I + (k+1) a^k b A + k a^{k-1} b^2 I^2 A^2 \dots (1) \end{aligned}$$

$$\text{Now } A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \text{ a zero matrix.}$$

(1) Becomes

$$(aI + bA)^{k+1} = a^{k+1} I + (k+1) a^k b A$$

Therefore the result is true for $n = k + 1$. Thus by Mathematical Induction we have

$$(aI + bA)^n = a^n I + n a^{n-1} b A$$

Question numbered: 4,5,6,7 try!

7. Matrix answers:

$$\text{Prove } I + A = (I - A) \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}, \text{ when } A = \begin{bmatrix} 0 & -\tan \frac{\alpha}{2} \\ \tan \frac{\alpha}{2} & 0 \end{bmatrix}$$

On the L.H.S.

$$\begin{aligned} I + A &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -\tan \frac{\alpha}{2} \\ \tan \frac{\alpha}{2} & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -\tan \frac{\alpha}{2} \\ \tan \frac{\alpha}{2} & 1 \end{bmatrix} \dots (1) \end{aligned}$$

On the R.H.S.

$$(I - A) \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$$

$$\begin{aligned}
& (I - A) \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -\tan \frac{\alpha}{2} \\ \tan \frac{\alpha}{2} & 0 \end{bmatrix} \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \\
&= \begin{bmatrix} 1 & \tan \frac{\alpha}{2} \\ -\tan \frac{\alpha}{2} & 1 \end{bmatrix} \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \\
&= \begin{bmatrix} \cos \alpha + \sin \alpha \tan \frac{\alpha}{2} & -\sin \alpha + \cos \alpha \tan \frac{\alpha}{2} \\ -\cos \alpha \tan \frac{\alpha}{2} + \sin \alpha & \sin \alpha \tan \frac{\alpha}{2} + \cos \alpha \end{bmatrix} \dots (2)
\end{aligned}$$

$$\begin{aligned}
&= \begin{bmatrix} 1 - 2 \sin^2 \frac{\alpha}{2} + 2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} \tan \frac{\alpha}{2} & -2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} + (2 \cos^2 \frac{\alpha}{2} - 1) \tan \frac{\alpha}{2} \\ -(2 \cos^2 \frac{\alpha}{2} - 1) \tan \frac{\alpha}{2} + 2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} & 2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} \tan \frac{\alpha}{2} + 1 - 2 \sin^2 \frac{\alpha}{2} \end{bmatrix} \\
&= \begin{bmatrix} 1 - 2 \sin^2 \frac{\alpha}{2} + 2 \sin^2 \frac{\alpha}{2} & -2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} + 2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} - \tan \frac{\alpha}{2} \\ -2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} + \tan \frac{\alpha}{2} + 2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} & 2 \sin^2 \frac{\alpha}{2} + 1 - 2 \sin^2 \frac{\alpha}{2} \end{bmatrix} \\
&= \begin{bmatrix} 1 & -\tan \frac{\alpha}{2} \\ \tan \frac{\alpha}{2} & 1 \end{bmatrix}
\end{aligned}$$

Thus, from (1) and (2), we get L.H.S. = R.H.S.

Question 13:

$$F(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If $F(x)F(y) = F(x+y)$, show that $F(x)F(y) = F(x+y)$.

Answer

$$F(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}, F(y) = \begin{bmatrix} \cos y & -\sin y & 0 \\ \sin y & \cos y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$F(x+y) = \begin{bmatrix} \cos(x+y) & -\sin(x+y) & 0 \\ \sin(x+y) & \cos(x+y) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned}
& F(x)F(y) \\
&= \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos y & -\sin y & 0 \\ \sin y & \cos y & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \cos x \cos y - \sin x \sin y + 0 & -\cos x \sin y - \sin x \cos y + 0 & 0 \\ \sin x \cos y + \cos x \sin y + 0 & -\sin x \sin y + \cos x \cos y + 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \cos(x+y) & -\sin(x+y) & 0 \\ \sin(x+y) & \cos(x+y) & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= F(x+y)
\end{aligned}$$

$$\therefore F(x)F(y) = F(x+y)$$

Question 14:

9. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = 0$, hence find A^{-1} .

$$\begin{aligned}
\text{Answer: } & \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} - 5 \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} + 7 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
& \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ 5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 8 - 15 + 7 & 5 - 5 + 0 \\ 5 - 5 + 0 & 3 - 10 + 7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\end{aligned}$$

Hence proved.

To find A^{-1} :

Pre-multiply $A^2 - 5A + 7I = 0$ by A^{-1}

$$A^{-1}(A^2 - 5A + 7I = 0) \Rightarrow$$

$$A^{-1}A^2 - 5A^{-1}A + 7A^{-1}I = 0$$

$$A^{-1}A A - 5I + 7A^{-1} = 0$$

$$IA - 5I + 7A^{-1} = 0$$

$$7A^{-1} = 0 + 5I - A$$

$$A^{-1} = \frac{1}{7}(5I - A)$$

$$\begin{aligned}
A^{-1} &= \begin{bmatrix} \frac{5}{7} & 0 \\ 0 & \frac{5}{7} \end{bmatrix} - \begin{bmatrix} \frac{3}{7} & \frac{1}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{bmatrix} \\
&= \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ \frac{1}{7} & \frac{3}{7} \end{bmatrix}
\end{aligned}$$

Similar to this problem is Q.No 11

(Q. No 9,10,11 try!)

12. The sum of three numbers is 6. If we multiply third number by 3 and add second number to it, we get

11. By adding first and third number we get double of the second number.

Solution: Let the 3 numbers be x, y and z .

$x + y + z = 6$; $y + 3z = 11$ and $x + z = 2y$ are the three equations hence collecting co-efficient of x, y, z and placed in a 3×3 matrix we get matrix A, write x, y, z as column matrix and values on RHS as another column matrix and write the following:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 11 \\ 0 \end{bmatrix}$$

This is the required matrix equation.

Try 12.b,c,d, 13,14 on your own.

15) (a) $(A+B)(A+B) \neq A^2 + 2AB + B^2$ (b) $(A+B)(A-B) \neq A^2 - B^2$. This is because of non-commutative property of product of two matrices.

$$(A+B)(A+B) = A \times A + A \times B + B \times A + B \times B$$

$$= A^2 + A \times B + B \times A + B^2 \neq A^2 + 2A \times B + B^2 \text{ as } A \times B \neq B \times A$$

Same reason can be applied to $(A+B)(A-B)$ to prove $(A+B)(A-B) \neq A^2 - B^2$.

Chapter -4: **Determinants:**

Mathematically the symbol $||$ or Δ is called *determinants*. A square matrix of order 1, 2, 3, 4 can be converted to determinant as:

$[\pi] = |\pi|$ a determinant of order 1

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$, here $ad - bc$ is the value of determinants. If a matrix is converted to determinants otherwise matrix represents an object or a number or a function for each elements.

How to evaluate a 3×3 determinant:

$$\text{Consider } \Delta = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

The value of this determined by $a \begin{vmatrix} e & f \\ g & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$

Here the expansion is by row 1 elements (a b c). The second order determinants is found by the 2×2 determinants by leaving the row and column in which the *first-row elements* are taken from. Also the sign

is alternatively taken as +, -, +. The 2×2 determinants are further evaluated as show above. This 2×2 is called minor of the prefixed row /column element.

We can expand by any row or column, and it is found that the value of determinant remains the same.

The minor of $a_{11} = \begin{vmatrix} e & f \\ f & i \end{vmatrix}$

The minor of $a_{12} = \begin{vmatrix} d & f \\ g & i \end{vmatrix}$

The minor of $a_{13} = \begin{vmatrix} d & e \\ g & h \end{vmatrix}$

Example: find the determinant of the matrices: (a) $\begin{bmatrix} 10 & 2 \\ -2 & 10 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & 4 \\ 3 & 2 & 1 \\ -1 & -2 & 0 \end{bmatrix}$

Answer: Let $A = \begin{bmatrix} 10 & 2 \\ -2 & 10 \end{bmatrix}$

The determinant of a is : $\det(A) = \begin{vmatrix} 10 & 2 \\ -2 & 10 \end{vmatrix} = 10 \times 10 - (2 \times (-2)) = 100 + 4 = 104$

Let $B = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 2 & 1 \\ -1 & -2 & 0 \end{bmatrix}$, the $\det(B) = \begin{vmatrix} 1 & 2 & 4 \\ 3 & 2 & 1 \\ -1 & -2 & 0 \end{vmatrix} = 1 \times \begin{vmatrix} 2 & 1 \\ -2 & 0 \end{vmatrix} - 2 \times \begin{vmatrix} 3 & 1 \\ -1 & 0 \end{vmatrix} + 4 \times \begin{vmatrix} 3 & 2 \\ -1 & -2 \end{vmatrix}$

$\det(B) = 1(2 \times 0 - 1 \times (-2)) - 2 \times (3 \times 0 - 1 \times (-1)) + 4 \times (3 \times (-2) - (2 \times 1))$
 $= 2 - (-2) + 4(-8) = 2 + 2 - 32 = -28$

Determinant is used to solve simultaneous equations.

We discuss here the method:

Consider a system of equation in two variables (say x, y):

$a_1x + b_1y = c$ and $a_2x + b_2y = c$

The solution exists only if the system is consistent: If $\frac{a_1}{b_1} \neq \frac{a_2}{b_2}$. Cases where system has no solution or infinite many solutions are discussed in class 9 mathematics.

There are 2 methods to solve such a system: 1) By Cramer's rule 2) By matrix inverse method

In class 12 (as per CBSE syllabus) we use method two.

How to solve a simultaneous equation using matrix inverse method?

1. Convert the given equation into a matrix equation $AX=B$ where A is matrix formed by co-efficient of x, y in the system of equations.
2. Find $\det A$. If $\det A \neq 0$, then the solution is given by $X = A^{-1}B$, where A^{-1} is inverse matrix of A .
3. Formula: $A^{-1} = \frac{1}{|A|} \text{adj}A$.

4. If $|A| = 0$, then we use (i) $\text{adj}A \times B = O$ the system may be either consistent or inconsistent according as the system have either infinitely many solutions or no solution. (ii) if $\text{adj}A \times B \neq O$, then solution of equation is inconsistent.

Problem on system of :

$$\text{Solve } 2x + 3y = 8, 3x - 2y = -1.$$

$$\text{The Matrix equation is } \begin{bmatrix} 2 & 3 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \end{bmatrix}.$$

$$A = \begin{bmatrix} 2 & 3 \\ 3 & -1 \end{bmatrix} \text{ and } \begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix} = 2(-1) - (3 \times 3) = -2 - 9 = -11.$$

$\det A \neq 0$, hence the system is **consistent** and has a **unique solution**.

$$\text{Now } A^{-1} = \frac{1}{-11} \begin{bmatrix} -1 & -3 \\ -3 & 2 \end{bmatrix}$$

$$\text{The solution matrix is } \begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}B = \frac{1}{-11} \begin{bmatrix} -1 & -3 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 8 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Hence $x = 1, y = 2$.

Verification: equation 1: $2(1) + 3(2) = 8$ and equation 2: $3(1) - 2(2) = -1$.

Similarly system of equations involving 3 variables x, y and z can be solved by matrix inversion method.

Example 2: System of equation involving 3 variables x, y and z .

$$a_1x + b_1y + c_1z = d_1, a_2x + b_2y + c_2z = d_2, \text{ and } a_3x + b_3y + c_3z = d_3$$

$$\text{The corresponding matrix equation is } \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

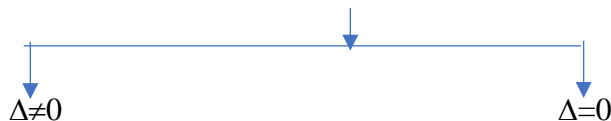
$$\text{This is of the form } AX = B \text{ where } A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}.$$

Criteria for consistency in 3 variables:

We create three determinants:

$$\Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}, \Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \text{ and } \Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Check value of Δ



Consistent system and has unique solution

check the value of Δ_1, Δ_2 and Δ_3

$$x = \frac{\Delta_1}{\Delta}, y = \frac{\Delta_2}{\Delta} \text{ and } z = \frac{\Delta_3}{\Delta}$$

At least one of Δ_1, Δ_2 and Δ_3 is not 0 all zero



Inconsistent system

put $z=t$

Solve the system in terms of t .

Properties of Determinants:

Determinants have some properties that are useful as they permit us to generate the same results with different

and simpler configurations of entries (elements).

- (a) **Reflection Property:** The determinant remains unaltered if its rows are changed into columns and the columns into rows.
- (b) **All-zero Property:** If all the elements of a row (or column) are zero, then the determinant is zero.
- (c) **Proportionality (Repetition) Property:** If the all elements of a row (or column) are proportional (identical) to the elements of some other row (or column), then the determinant is zero.
- (d) **Switching Property:** The interchange of any two rows (or columns) of the determinant changes its sign.
- (e) **Scalar Multiple Property:** If all the elements of a row (or column) of a determinant are multiplied by a non-zero constant, then the determinant gets multiplied by the same constant.

(f) **Sum property:**
$$\begin{vmatrix} a_1 + b_1 & c_1 & d_1 \\ a_2 + b_2 & c_2 & d_2 \\ a_3 + b_3 & c_3 & d_3 \end{vmatrix} = \begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix} + \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}$$

(g) **Property of Invariance:**

a determinant remains unaltered under an operation of the form $C_i \rightarrow C_i + aC_j + bC_k$, where $j, k \neq i$, or an operation of the form $R_i \rightarrow R_i + aR_j + bR_k$, where $j, k \neq i$

(h) **Factor Property:** If a determinant D becomes zero when we put $x = a$, then $(x - a)$ is a factor of D .

• **Triangle Property:** If all the elements of a determinant above or below the main diagonal consist of zeros, then the determinant is equal to the product of diagonal elements. That is,

$$\begin{vmatrix} a_1 & 0 & 0 \\ b_1 & b_2 & 0 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 b_2 c_3 \text{ also } \begin{vmatrix} a_1 & a_2 & a_3 \\ 0 & b_2 & b_3 \\ 0 & 0 & c_3 \end{vmatrix} = a_1 b_2 c_3.$$

(i) Determinant of cofactor matrix: $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ then Determinant of cofactor matrix is

$$\Delta_1 = \begin{vmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{vmatrix}, \text{ where } C_{ij} \text{ denotes the cofactor of the element } a_{ij} \text{ in } D. \text{ Result: } \Delta_1 = \Delta^2.$$

Problems:

1. Solve the following equations by matrix method $x + y + z = 9$, $2x + 5y + 7z = 52$, $2x + y - z = 0$. (Ans: $x = 1$, $y = 3$, $z = 5$)
2. For what value of k will the system equation have non-trivial solutions. Also find all the solutions of the system for that value of k . System is : $x + y - kz = 0$; $3x - y - 2z = 0$; $x - y + 2z = 0$

Some important results:

1. The lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$ (*only two variables in this system*) are

$$\text{concurrent if } \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

2. The general equation of a **conic** is $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ this represents a pair of straight lines if

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

3. Area of triangle whose vertices are (x_r, y_r) , where $r = 1, 2$ and 3 then $\text{Area} = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \text{sq. units}$

4. If this area = 0, then the points (x_r, y_r) , where $r = 1, 2$ and 3 are collinear.

5. Equation of a straight line passing through (x_1, y_1) and (x_2, y_2) is given by the determinant $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} =$

0 .

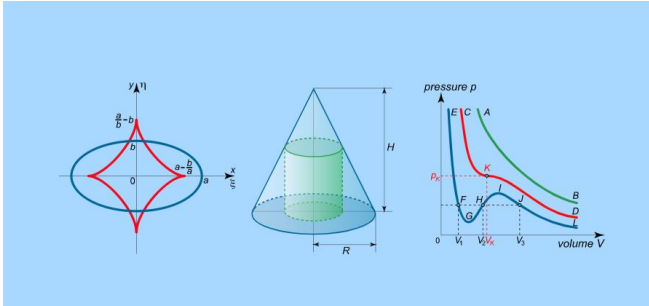
We shall see properties on applying calculus (*differentiation and integrations*) in determinants consisting of real valued functions.

Mathematics

Class 12

Volume -I

Calculus



By B. Harendran, M.Sc. B.Ed

Content:

Differentiation

Methods of finding derivatives of a function

Application of derivative of a function

Integration

Application of integration

Differential Equation

Differentiation

To understand derivative of a function one must have the basic concepts of finding limit of a function. The following are formulae of limits:

The derivative of a real valued non zero function is defined as follows:

Let $y = f(x)$ be a real valued function, let a small increment in x be Δx then $\lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$ is called derivative of $f(x)$

w.r.t x .

Different notations: $f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$ also if $y = f(x)$, then $\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = f'(x)$

$\frac{dy}{dx}$ is also denoted as y_1 , when $y=f(t)$ is a function of t =time, then $\frac{dy}{dt} = \dot{y}$.

When the derivative or differentiation of a function is found by the above definition, we call it as **first principal method or ab-initio method**.

Once we find the derivative of all kinds of function (refer chapter 1, relation & function), they can be used as formula to do higher order thinking problems, complex problems.

Here are the basic kinds of functions whose derivatives have been found by ab-initio method:

Basic functions	Derivative $\frac{dy}{dx}$	Notes
$Y = c$ (constant)	0	All types of constants used in Mathematics, Physics, Chemistry, Economics
$Y = x^n$, x is a real variable, $n \in \mathbb{R}$	$\frac{dy}{dx} = nx^{n-1}$	
$Y = e^x$	$\frac{dy}{dx} = e^x$	e is an Euler constant $2 < e < 3$
$Y = \log x$, $x > 0$	$\frac{dy}{dx} = \frac{1}{x}$	$x > 0$

$Y = cf(x)$	$\frac{dy}{dx} = cf'(x)$	C is a constant
$Y = \sin x$	$\frac{dy}{dx} = \cos x$	$x \in \mathbb{R}$, measured in radians
$Y = \cos x$	$\frac{dy}{dx} = -\sin x$	$x \in \mathbb{R}$, measured in radians
$Y = \tan x$	$\frac{dy}{dx} = \sec^2 x$	$x \in \mathbb{R}$, measured in radians
$Y = \operatorname{cosec} x$	$\frac{dy}{dx} = -\operatorname{cosec} x \cot x$	$x \in \mathbb{R} - n\pi, n \in \mathbb{Z}$
$Y = \sec x$	$\frac{dy}{dx} = \sec x \tan x$	$x \in \mathbb{R} - (2n+1)\frac{\pi}{2}, n \in \mathbb{Z}$
$Y = \cot x$	$\frac{dy}{dx} = -\operatorname{cosec}^2 x$	$x \in \mathbb{R} - n\pi, n \in \mathbb{Z}$
$Y = a^x$	$\frac{dy}{dx} = \log_a(a^x)$	$x \in \mathbb{R}$
$y = \sin^{-1} x$	$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$	$-1 < x < 1$
$y = \cos^{-1} x$	$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$	$-1 < x < 1$
$y = \tan^{-1} x$	$\frac{dy}{dx} = \frac{1}{1+x^2}$	$x \in \mathbb{R}$
$y = \operatorname{cosec}^{-1} x$	$\frac{dy}{dx} = \frac{-1}{ x \sqrt{x^2-1}}$	$(-\infty, -1) \cup (1, \infty)$
$y = \cot^{-1} x$	$\frac{dy}{dx} = \frac{-1}{1+x^2}$	$x \in \mathbb{R}$
$Y = \sec^{-1} x$	$\frac{dy}{dx} = \frac{1}{ x \sqrt{x^2-1}}$	$(-\infty, -1) \cup (1, \infty)$
$Y = f(x) \pm g(x)$	$\frac{dy}{dx} = f'(x) \pm g'(x)$	
$Y = f(x) \times g(x)$	$\frac{dy}{dx} = f(x) \times g'(x) + f'(x) \times g(x)$	This is product rule. If $f(x) = u$, $g(x) = v$ then formula is easy to memorize as $uv' + u'v$
$y = \frac{f(x)}{g(x)} = \frac{u}{v}$	$\frac{dy}{dx} = \frac{u'v - uv'}{v^2}$	The denominator function $g(x) = v$ cannot be zero.
$y = f \circ g(x) = f(g(x))$	$\frac{dy}{dx} = f'(g(x)) \times g'(x)$	This is derivative of composition of functions. A basis for chain rule.

Methods of differentiation:

1. Chain rule

2. **Parametric form differentiation**
3. **Implicit function – differentiation**
4. **Logarithmic - differentiation**

Chain Rule:

Some functions are functions of another function(s). Example: $y = \sin(\cos x)$, $e^{\sin x}$, $\log(ax^2+bx+c)$ etc., Method of differentiation is :

1. Let $g(x) = \cos x$ in $y = \sin(\cos x)$, then $y = \sin(g(x))$, $\frac{dy}{dx} = \cos(g(x)) \times g'(x)$

$$= \cos(\cos x) \times (-\sin x). \text{ (derivative } \cos x = -\sin x)$$

2. $Y = e^{\sin x}$, let $g(x)$ or $t = \sin x$.

Now differentiate $y = e^t$, differentiate w.r.t t then $\frac{dy}{dt} = e^t$. As y is a function of x , the derivative becomes $\frac{dy}{dx} =$

$$\frac{dy}{dt} \times \frac{dt}{dx} = e^t \times \cos x \text{ as } \frac{dt}{dx} = \cos x.$$

Expressing in terms of x , we get $\frac{dy}{dx} = e^{\sin x} \times \cos x$.

3. $Y = \log(ax^2+bx+c)$

Let $t = ax^2+bx+c$, then $\frac{dt}{dx} = 2ax + b$

Now $y = \log_e t$ then $\frac{dy}{dt} = \frac{1}{t}, t > 0$.

Thus $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} \Rightarrow \frac{1}{t} \times 2ax + b$.

Rewriting in terms of t , $\frac{dy}{dx} = \frac{1}{ax^2+bx+c} \times 2ax + b = \frac{2ax+b}{ax^2+bx+c}$.

HOT problems will be of the form : $y = uv$, or $\frac{u}{v}, v \neq 0$ where u and v will involve chain rule when u' and v' are found.

Example: differentiate w.r.t x , $f(x) = e^{\tan x} \times [5(\sin^2 x) + 7\cos^3 x]$.

Here $u(x) = e^{\tan x}$ and $v(x) = 5(\sin^2 x) + 7\cos^3 x$.

$$f'(x) = u'(x)v(x) + u(x)v'(x)$$

derivative of $u(x)$, $v(x)$ uses chain rule as follows:

$$u(x) = e^{\tan x} = e^t \text{ and } t = \tan x, u'(x) = \frac{du}{dt} \times \frac{dt}{dx} = e^t \times \sec^2 x.$$

Therefore $u'(x) = e^{\tan x} \sec^2 x$.

Similarly, $v'(x) = \frac{d}{dx}(5\sin^2 x) + \frac{d}{dx}(7\cos^3 x)$

$$= 5 \frac{du}{dt} \times \frac{dt}{dx} = 5(2t \times \cos x), \text{ where } u=t^2 \text{ and } t=\sin x + 7 \frac{ds}{dw} \times \frac{dw}{dx} = 7(3t^2 \times (-\sin x)), \text{ here } s=w^3 \text{ and}$$

$w = \cos x$.

$V'(x) = 10t + 5\cos x + 21w^2 + (-7\sin x)$, substituting t, w in terms of x we get final solution as:

$$f'(x) = u'(x)v(x) + u(x)v'(x)$$

$$= (e^{\tan x} \sec^2 x)[5(\sin^2 x) + 7\cos^3 x] + e^{\tan x} \times \{10\sin x + 5\cos x + 21\cos^2 x + (-7\sin x)\} \dots \text{Answer.}$$

To get confidence in differentiation using chain rule in combination with product rule, quotient rules, practice as many problems of this type. *Do a self-test every weekend.*

Parametric form differentiation

It is a function $f(x, y)$, where $x=g(t)$ and $y=h(t)$, x and y are related (functions) of a common variable t . The following formulae are used in differentiation of parametric form:

Let $x=g(t)$ and $y=h(t)$

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$

The second derivative of a parametric functions is found by using the formula: $\frac{d^2y}{dx^2} = \frac{d}{dt} \left(\frac{dy}{dx} \right) \times \frac{dt}{dx}$. Note:

$\frac{dy}{dx}$ is a function of t . Hence it is differentiated w.r.t t and multiplied by the reciprocal of $\frac{dx}{dt}$.

Common error: Student find 2nd derivative of $x=g(t)$ w.r.t t and $y=h(t)$ and find $\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} \div \frac{d^2x}{dt^2}$, which is completely wrong!

Example:

1. The parametric equation : $x = a\cos\theta$, $y = a\sin\theta$, represent the equation of circle $x^2 + y^2 = a^2$.

The derivative $\frac{dy}{dx}$ is found as:

$$\frac{dx}{d\theta} = -a\sin\theta, \frac{dy}{d\theta} = a\cos\theta \text{ then } \frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \frac{a\cos\theta}{-a\sin\theta} = -\cot\theta \text{ (note here } \theta \text{ is the parameter)}$$

2. The conic: parabola: $y^2 = 4ax$ has parametric equation: $x = at^2$, $y = 2at$. The derivative is found as $\frac{dx}{dt} = 2at$ and $\frac{dy}{dt} = 2a$

$$\text{Therefore } \frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{2a}{2at} = \frac{1}{t}$$

Problems:

- Find the derivative $\frac{dy}{dx}$, of the parametrically defined curve $x(t)=2\cos(4t)$, $y = \sin(4t)$. Ans: $-\frac{1}{2}\cot(4t)$
- $x=t^2$, $y=t^3$. Ans: $3t/2$, $t \neq 0$.
- $x=e^{2t}$, $y=e^{3t}$ Ans: $\frac{3}{2}t$
- $x = \sin^2t$, $y = \cos^2t$. Ans: -1 , $t \neq \frac{n\pi}{2}$, $n \in \mathbb{Z}$.
- $x=2t^2+t+1$, $y = 8t^3+3t^2+5$, Ans: $6t$
- $x = \sqrt{1-t^2}$, $y = \arcsint$. Ans: $-\frac{1}{t}$, $|t| < 1$, $t \neq 0$.

Find the second derivative of the parametric equation of a parabola, which has x -axis as its axis, focus lying on x -axis, where focus is $(a, 0)$.

Answer: The conic: parabola: $y^2 = 4ax$ has parametric equation: $x = at^2$, $y = 2at$.

The derivative $\frac{dy}{dx} = \frac{1}{t}$, $t \neq 0$.

$$\begin{aligned} \text{The second derivative is } \frac{d^2y}{dx^2} &= \frac{d}{dt} \left(\frac{1}{t} \right) \times \frac{dt}{dx} \\ &= \frac{-1}{t^2} \times \frac{1}{2at} \\ &= \frac{-1}{2at^3}. \end{aligned}$$

Derivative of implicit functions:

An implicit function is a function that can be expressed as $f(x, y) = 0$. i.e., it cannot be easily solved for 'y' (or) it cannot be easily got into the form of $y = f(x)$.

Examples: $x^2 + 4xy + y^2 = 0$

How to differentiate this *implicit function*:

we differentiated every term with respect to x by considering y as a function of x , and this type of differentiation is called implicit differentiation. But for some functions like $xy + \sin(xy) = 0$, writing it as an explicit function ($y=f(x)$) is not possible. In such cases, only implicit differentiation is the way to find the derivative.

Note treating $f(y)$ and differentiating w.r.t x , first differentiate with respect to y and multiply by $\frac{dy}{dx}$.

Differentiate

$$x^2 + 4xy + y^2 = 0$$

$$\frac{d}{dx}x^2 + 4\frac{d}{dx}(xy) + \frac{d}{dx}y^2 = 0$$

$$2x + 4\left(x\frac{d}{dx}y + \frac{d(x)}{dx}y\right) + 2y\frac{dy}{dx} = 0 \quad (\text{product rule applied in 2nd term})$$

Collect all terms containing $\frac{dy}{dx}$ on left side and post other terms to right side of =.

$$4x\frac{dy}{dx} + 2y\frac{dy}{dx} = -2x - 4y$$

$$(4x + 2y)\frac{dy}{dx} = -2x - 4y$$

$$\text{Therefore } \frac{dy}{dx} = \frac{-2(x+y)}{2(2x+y)} = -\frac{x+y}{2x+y}. \text{ Answer.}$$

Logarithmic - differentiation

When a function is made up of products/quotients of many function, we use logarithmic function method.

Concept formula: Let $g(x) = \log_e\{f(x)\}$, the derivative $\frac{d}{dx}g(x)$ is found as $g'(x) = \frac{1}{f(x)}f'(x)$.

Examples:

1. Find the derivative of $\log(\sin x)$?

Answer: $y = \log(\sin x)$

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{\sin x} \frac{d}{dx}(\sin x) \\ &= \frac{\cos x}{\sin x} = \cot x\end{aligned}$$

2. Find the derivative of $y = (x^3 + 3x^2 - 5x + 7)\left(x + \frac{1}{x}\right)(e^{\tan x})$

Answer: Taking logarithm on both side we get

$$\log y = \log(x^3 + 3x^2 - 5x + 7) + \log\left(x + \frac{1}{x}\right) + \log(e^{\tan x})$$

Note: we use the fundamental property of $\log(abc) = \log a + \log b + \log c$

Now differentiating we get

$$\begin{aligned}\frac{1}{y} \frac{dy}{dx} &= \frac{1}{(x^3 + 3x^2 - 5x + 7)}(x^3 + 3x^2 - 5x + 7)' + \frac{1}{\left(x + \frac{1}{x}\right)}\left(x + \frac{1}{x}\right)' + \frac{1}{(e^{\tan x})}(e^{\tan x})' \\ &= \frac{3x^2 + 6x - 5}{x^3 + 3x^2 - 5x + 7} + \frac{x + \frac{1}{x}}{1 - \frac{1}{x^2}} + \frac{e^{\tan x} \times \sec^2 x}{e^{\tan x}}\end{aligned}$$

$$\text{Therefore } \frac{dy}{dx} = y \times \left(\frac{3x^2 + 6x - 5}{x^3 + 3x^2 - 5x + 7} + \frac{x + \frac{1}{x}}{1 - \frac{1}{x^2}} + \frac{e^{\tan x} \times \sec^2 x}{e^{\tan x}} \right).$$

Problems based on $y = f(x)^{g(x)}$

Method: Take logarithm on both side we find: $\log y = g(x)\log \{f(x)\}$ using $\log(m^n) = n \log m$.

Differentiate LHS as $\frac{1}{y} \frac{dy}{dx}$ and use product rule *differentiation* for right hand side.

Example:

Find the derivative of x^x

Answer: let $y = x^x$, take logarithm on both side we get $\log y = x \log x$. On differentiating we find $\frac{1}{y} \frac{dy}{dx} = x(\log x)' + \log x \times \frac{d(x)}{dx} \Rightarrow \frac{dy}{dx} = y \left(x \frac{1}{x} + \log x \times 1 \right) = y(1 + \log x)$.

Find the derivative of $x^x + y^y$.

Error: If $y = x^x + y^y$, taking logarithm we find $\log y = \log(x^x + y^y)$. Now error made on *right hand side* is $\log(x^x + y^y) \neq \log x^x + \log y^y$, this wrong as $\log(m+n) \neq \log m + \log n$. Hence $\log(x^x + y^y) \neq \log x^x + \log y^y$.

How to differentiate?

Consider $u = x^x$ and $v = y^y$, $\frac{du}{dx} = u(1 + \log x)$ and $\frac{dv}{dx}$ is found as follows:

As $v = y^y$, $\log v = y \log y$.

Differentiating we get $\frac{dv}{dx} = v \left(x \frac{1}{v} \frac{dy}{dx} + \log y \times 1 \right)$

Now adding $\frac{du}{dx} + \frac{dv}{dx} = x^x(1 + \log x) + y^y \left(x \frac{dy}{y dx} + \log y \right)$.

There are problems which are of the type $y = f(x)^{f(x)^{f(x)^{\dots\infty}}}$. This can be differentiated by considering the equivalent expression $y = f(x)^y$.

Example : Find the derivative of the *infinite exponential function* $x^{x^{x^{\dots\infty}}}$.

Let $y = x^{x^{x^{\dots\infty}}}$, taking the equivalent equation as $y = x^y$.

Taking logarithm, we get $\log y = y \log x$.

Differentiating we get $\frac{1}{y} \frac{dy}{dx} = \frac{y}{x} + \log x \frac{dy}{dx}$.

Rearranging and simplifying we get $\left(\frac{1}{y} + \log x \right) \frac{dy}{dx} = \frac{y}{x}$.

Therefore $\frac{dy}{dx} = \frac{y/x}{\left(\frac{1}{y} + \log x \right)}$.

Higher order derivatives:

When the function $f(x)$ is differentiated first time it is first order derivative represented by $f'(x)$ or $\frac{dy}{dx}$ or y_1 .

When $f'(x)$ is differentiated (if exist) again it is second order derivative denoted by $f''(x)$ or $\frac{d^2y}{dx^2}$ or y_2 . This way one can differentiate to get 3rd order to n th order derivatives.

Example: find the 4th derivative of $y = x^4$. $y_1 = 4x^3$, $y_2 = 12x^2$, $y_3 = 24x$, $y_4 = 24$.

Example: let $y = e^x$.

$$y_1 = e^x$$

$$y_2 = e^x, \dots y_n = e^x.$$

There are identities derived using higher order derivatives.

1. If $y = 3e^{2x} + 2e^{3x}$, prove that $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$.
2. If $y = A\sin x + B\cos x$, then $\frac{d^2y}{dx^2} + y = 0$.
3. If $y = \sin^{-1} x$, prove that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0$.
4. If $e^x(x+1) = 1$, find a relation between $\frac{d^2y}{dx^2}$ and $\frac{dy}{dx}$.
5. If $y = e^{-(2/3)x}[A \sin(5/3)x + B \cos(5/3)x]$, verify that equation $9y'' + 12y' + 29y = 0$.

Theorems without proof:

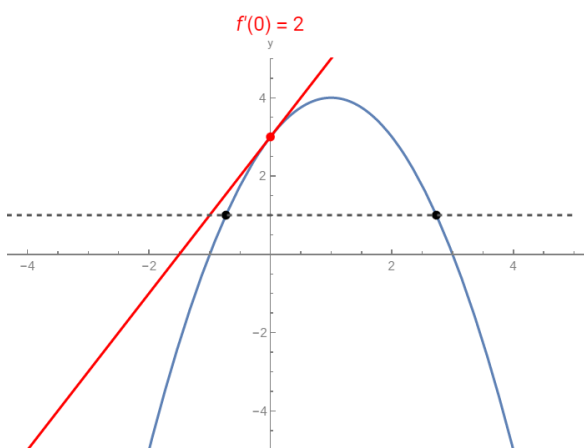
1. **Rolle's theorem**
2. **Mean Value theorem.**

Rolle's theorem:

Statement: Let $f: [a, b] \rightarrow \mathbb{R}$ be a continuous on $[a, b]$ and differentiable on (a, b) such that $f(a) = f(b)$, then there exists some c in (a, b) , such that $f'(c) = 0$. Note $a, b \in \mathbb{R}$.

Geometrical meaning of Rolle's theorem:

The geometry of $f'(x)$ is slope of tangent at a point on the graph of $f(x)$.



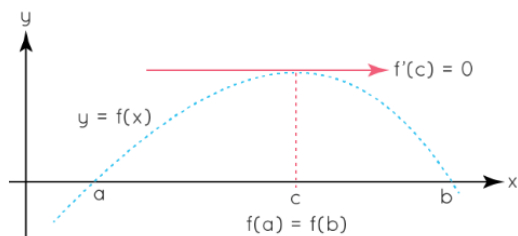
In this graph at $(0,2)$, a tangent is drawn the slope the tangent is $f'(x)$.

$$F(x) = -x^2 + 2x + 3$$

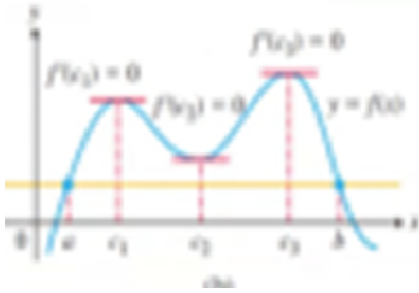
$$F'(x) = -2x + 3$$

$$F'(0) = 3 \Rightarrow \tan \psi = 3, \text{ where } \psi \text{ is the angle made by the line (tangent) with the } x\text{-axis.}$$

Now in the following graph, the tangent is parallel to x -axis, $\tan \psi = 0$



The geometrical meaning of *Rolle's theorem* is the slope of tangent is zero, tangent is parallel to x -axis. The behavior of the graph of $f(x)$ is the curve bends at $f'(x) = 0$.



In this graph There are three points $(c_1, f(c_1))$, $(c_2, f(c_2))$ and $(c_3, f(c_3))$ where the graph has a turning point and the tangents at them are parallel to x-axis.

This obeys Rolle's theorem.

The Mean Value Theorem is one of the most important theorems in Introductory Calculus, and it forms the basis for proofs of many results in subsequent and advanced Mathematics courses. The [history](#) of this theorem begins in the 1300's with the Indian Mathematician [Parameshvara](#), and is eventually based on the academic work of Mathematicians [Michel Rolle](#) in 1691 and [Augustin Louis Cauchy](#) in 1823.

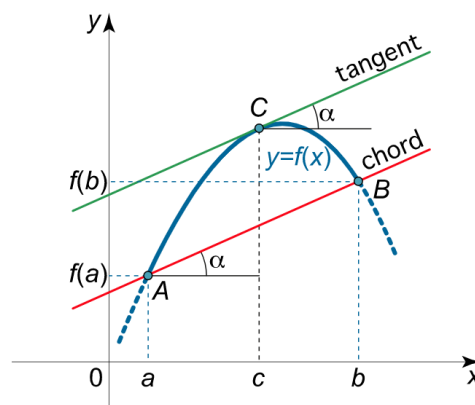
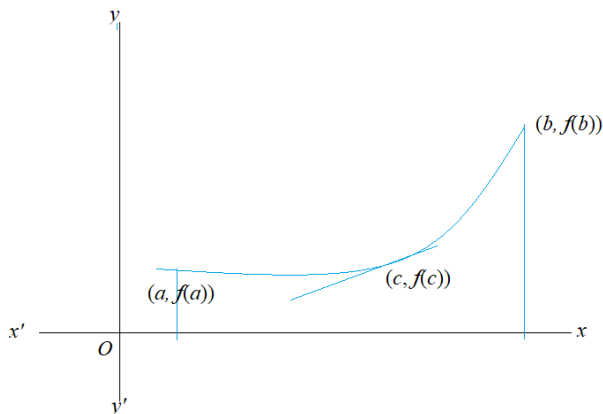
The formal statement of this theorem together with an illustration of the theorem will follow. I will also state [Rolle's Theorem](#), which is used in the proof the Mean Value Theorem. Both theorems are given without proof, and all subsequent problems here will be referencing only the Mean Value Theorem. All functions are assumed to be real-valued.

click on link (blue highlighted) to get more info.

Mean Value Theorem: If $f: [a, b] \rightarrow \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b) then there exists some c in (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Note: **Mean Value Theorem** is an extension of Rolle's theorem. When $f(a) \neq f(b)$, we apply MVT (**Mean Value Theorem**).

Geometrical Meaning of Mean Value Theorem:



In the above graph, the curve is continuous, and tangent at c is not parallel to x-axis, hence at $(c, f(c))$ the slope of tangent is given by $f'(c) = \frac{f(b)-f(a)}{b-a}$. The line touching is parallel to the line joining $A(a, f(a))$ to $B(b, f(b))$. This line AB is called secant.

Any line parallel to line AB can be found by MVT theorem.

Problem on Rolle's theorem:

1. Verify Rolle's theorem for the function $f(x) = x^2+2x$, $a=-2$ and $b=2$.
2. Verify Rolle's theorem for $f(x) = x^2+2x - 8$, $x \in [-4, 2]$.
3. Examine if Rolle's theorem is applicable.

a) $F(x)=[x]$ for $x \in [5, 9]$

b) $F(x) = x^2-1, x \in [1, 2]$

4. Can you discuss about the converse of Rolle's theorem from the above problems?

Conditions of Rolle's theorem

1. $f(x)$ is continuous at (a, b)

2. $f(x)$ is derivable at (a, b)

3. $f(a) = f(b)$

If all 3 conditions satisfied then

there exist some c in (a, b)

such that $f'(c) = 0$

5. If $f: [-5, 5] \rightarrow \mathbb{R}$ is a differentiable function and if $f'(x)$ does not vanish anywhere, then prove $f(-5) \neq f(5)$.

Note: In proving Rolle's theorem, (a) prove continuity and differentiability of given function (b) Verify $f(a) = f(b)$ a, b are the end point x values given in domain. Then find value of c such that $f'(c) = 0$ and check c belong to the given domain. If not Rolle's theorem is not valid or false.

Answers:

1) $f(x) = x^2+2x$, domain is $[-2, 2]$

i) As $f(x)$ is an algebraic function it is **continuous function and is differentiable** in $(-2, 2)$.

ii) $F(-2) = (-2)^2 + 2(-2) = 8$ and $f(2) = (2)^2 + 2(2) = 8$

Hence $f(-2) = f(2)$.

iii) Now $f'(x) = 2x+2$

$f'(x) = 0 \Rightarrow 2x+2 = 0$, hence $x = -1$,

$x = -1 \in (-2, 2)$.

Hence Rolle's theorem is proved and verified.

2) $f(x) = x^2+2x - 8, x \in [-4, 2]$

i) As $f(x)$ is polynomial (quadratic) function it is **continuous and differentiable** in $(-4, 2)$

ii) $F(-4) = (-4)^2 + 2(-4) - 8 = 16-8-8 = 0$

$F(2) = (2)^2 + 2(2) - 8 = 8 - 8 = 0$

Hence $f(-4) = f(2)$

iii) Now $f'(x) = 2x+2$

$f'(x) = 0 \Rightarrow 2x+2 = 0 \Rightarrow x = -1$ which belongs to $(-4, 2)$.

Therefore, Rolle's theorem is verified.

3) $F(x) = [x]$, the greatest integer function.

i) $Y=[x]$ is not a continuous function in $[5,9]$

There fore $f(x)$ is not differentiable.

ii) $F(5) = [5] = 5$ and $f(9) = [9] = 9$, hence $f(5) \neq f(9)$

Hence Rolle's theorem is false for this function.

4) In the case of $y=[x]$, $x \in [a, b]$, the range is an integer, then $\frac{dy}{dx} = 0$, the value of c can be any from (a, b) . but

$f(a) \neq f(b)$, the converse cannot be true.

Converse of Rolle's Theorem

If $f [a, b] \rightarrow R$

for some $c \in [a, b]$

for which $f'(c) = 0$ then

(i) $f(a) = f(b)$

(ii) f is continuous at $[a, b]$

(iii) & Differentiable at $[a, b]$

5) Given $f(x) : [-5, 5] \rightarrow R$ is a differentiable function the for any value c in $[-5, 5]$ we have $f'(c) = \frac{f(5) - f(-5)}{(-5) - 5} \neq 0$, by *MVT*.

$$f(5) - f(-5) \neq 0$$

Hence $f(5) \neq f(-5)$.

MVT problems and answers:

1) Check the validity of Mean value theorem for the function $f(x) = x^2 - 3x + 5$ on the interval $[1, 4]$. Find a point c satisfying the condition of the theorem.

Solution.

The given quadratic function is continuous and differentiable on the entire set of real numbers. Hence, we can apply **mean value theorem**. The derivative of the function has the form

$$f'(x) = (x^2 - 3x + 5)' = 2x - 3.$$

Find the coordinates of the point c :

$$f'(c) = \frac{f(b) - f(a)}{b - a} \Rightarrow \frac{((4)^2 - 3(4) + 5) - ((1)^2 - 3(1) + 5)}{4 - 1}$$

Therefore $2c - 3 = 2$

$$C = 5/2 \in [1, 4].$$

Therefore *MVT* is valid for the function $f(x) = x^2 - 3x + 5$ on the interval $[1, 4]$.

2) Find a point c satisfying Mean Value Theorem for the function $f(x) = \sqrt{x + 4}$ on the interval $[0, 5]$.

Solution.

The function is continuous on the closed interval $[0, 5]$ and differentiable on the open interval $(0, 5)$, so the *MVT* is applicable to the function.

The derivative has the form

$$f'(x) = (\sqrt{x + 4})' = \frac{1}{2\sqrt{x + 4}}.$$

Find the coordinates of the point c :

$$f'(c) = \frac{f(b) - f(a)}{b - a}, \Rightarrow \frac{1}{2\sqrt{c + 4}} = \frac{\sqrt{5 + 4} - \sqrt{0 + 4}}{5 - 0}, \Rightarrow \frac{1}{2\sqrt{c + 4}} = \frac{1}{5}, \Rightarrow \sqrt{c + 4} = \frac{5}{2}, \Rightarrow c + 4$$

It can be seen that the point $c = 2.25$ belongs to the open interval $(0, 5)$.

3) The position of a particle is given by the function of *time* $s(t) = 2t^2 + 3t - 4$. Find the time $t = c$ in the interval $0 \leq t \leq 4$ when the instantaneous velocity of the particle equals to its average velocity in this interval.

Solution.

The function $s(t)$ satisfies the conditions of the Mean Value Theorem, so we can write $s'(c) = \frac{s(b)-s(a)}{b-a}$, where $a=0$,

$b = 4$.

Take the derivative:

$$s'(t)=(2t^2+3t-4)'=4t+3.$$

Substituting this in the formula above, we get

$$4c+3=\frac{s(4)-s(0)}{4-0} \quad 40-(-4)/4,\Rightarrow 4c+3=11,\Rightarrow 4c=8,\Rightarrow c=2.$$

Answer: $2 \in (0,4)$

4) Examine the applicability of Mean value theorem.

Answer:

Physical interpretation

The mean value theorem has also a clear physical interpretation. If we assume that $f(t)$ represents the position of a body moving along a line, depending on the time t , then the ratio of $\frac{f(b)-f(a)}{b-a}$

is the average velocity of the body in the period of time $b-a$. Since $f'(t)$ is the instantaneous velocity, this theorem means that there exists a moment of time c , in which the instantaneous speed is equal to the average speed.

Mean value theorem has many applications in mathematical analysis, computational mathematics and other fields. Let us further note two remarkable corollaries.

Corollary 1

In a particular case when the values of the function $f(x)$ at the endpoints of the segment $[a, b]$ are equal, i.e. $f(a)=f(b)$, the mean value theorem implies that there is a point $c \in (a, b)$ such that

$f'(c) = \frac{f(b)-f(a)}{b-a}$, that is, we get Rolle's theorem, which can be considered as a special case of Mean value theorem.

Corollary 2

If the derivative $f'(x)$ is zero at all points of the interval $[a, b]$, then the function $f(x)$ is constant on this interval. Indeed, for any two points x_1 and x_2 in the interval $[a, b]$, there exists a point $c \in (a, b)$ such that $\frac{f(x_2)-f(x_1)}{b-a} = 0$.

$$f(x_2)-f(x_1)=f'(c)(x_2-x_1)=0 \cdot (x_2-x_1)=0.$$

Examples of application of MVT:

1. A car starts from rest and drives a distance of 10 km in 30 min. Use the mean value theorem to show that the car attains a speed of 20 km/hr at some point(s) during the interval. Ans: 20 Km/Hr.

2. Suppose that $f(x)$ is a differentiable function for all x . If $f'(x) \leq 7$ for all x and $f(2) = -4$, what is the maximum value of $f(5)$? Ans: 17

3. Use the mean value theorem to prove that $\ln(x+1) < x$ for $x > 0$.

Proof:

Suppose that our function was $f(t) = \ln(t + 1) - t$. Note that t is just a dummy variable as we will be using x in our interval of choice. Now, given that $f(t)$ is defined, continuous, and differentiable over the interval $[0, x]$, by the mean value theorem, we see that

$$f'(c) = \frac{f(x) - f(0)}{x - 0} \text{ for } c \in (0, x)$$

$$\frac{1}{c+1} - 1 = \frac{\ln(x+1) - x}{x}.$$

By observation, we know that $c + 1 > 1$, so it follows that $\frac{1}{c+1} < 1$. Thus, $\frac{1}{c+1} - 1 < 0$.

This tells us that $\frac{\ln(x+1) - x}{x} < 0 \implies \ln(x+1) < x$ for $x > 0$. \square

5) Suppose we know that $f(x)$ is continuous and differentiable on the interval $[-7, 0]$, that $f(-7) = -3$ and that $f'(x) \leq 2$. What is the largest possible value for $f(0)$? Ans: 11

6) Show that $f(x) = x^3 - 7x^2 + 25x + 8$ has exactly one real root. *Hint: find $f'(c) = 0$, if c is complex, derivative is a quadratic, hence 2 non real roots. Therefore only one real root exists.*

Application of derivative of real valued functions:

1. **Rate of change of quantities** (*deleted in CBSE*)

2. **Increasing and decreasing function.**

3. **Tangent and Normal** (*deleted in CBSE*)

4. **Approximation** (*deleted in CBSE*)

5. **Maxima and Minima.**

1. **Rate of change of quantities:** If a quantity y varies with another quantity x , and $y = f(x)$, then $\frac{dy}{dx}$ represent the rate of change of y with respect to x and $f'(x_0)$ is rate of change of y at x_0 .

2. A function is said to be (a) **increasing on the interval (a, b)** if $x_1 < x_2$ in $(a, b) \implies f(x_1) < f(x_2)$ for all $x_1, x_2 \in (a, b)$.

Alternatively, if $f'(x) \geq 0$ for each x in (a, b) (i) decreasing on (a, b) if $x_1 < x_2$ in $(a, b) \implies f(x_1) > f(x_2)$ for all $x_1, x_2 \in (a, b)$. (ii) **constant** in (a, b) , if $f(x) = c$ for all $x \in (a, b)$ where c is a constant.

3. A function is neither increasing nor decreasing is a **monotonic function**

4. Give a real valued function $f(x)$ and the solution(s) of $f'(x) = 0$ gives the x co ordinate of **critical points**.

Other names for critical points are i) Stationary points 2) turning points (*used in sketching*)

5. The value of $f'(x)$ at a point x_0 , $f'(x_0)$ is the slope of a tangent to the graph (*curve drawn using $y = f(x)$)* at $(x_0, f(x_0))$.

6. When $f'(x) = 0$, then the tangent at $(x_0, f(x_0))$ is **parallel** to the x - axis.

7. If $f(x_0)$ does not exist at $(x_0, f(x_0))$, then the tangent at that point is parallel to y - axis. *Note the slope of tangent is undefined, hence angle made by the tangent is 90° .* The normal (a line perpendicular to the tangent) is parallel to the x - axis.

8. The equation of tangent at $(x_0, f(x_0))$ is $y - y_0 = \left(\frac{dy}{dx}\right)_{x=x_0} (x - x_0)$ where $y_0 = f(x_0)$.

9. Let $m = \left(\frac{dy}{dx}\right)_{x=x_0}$, then the equation of Normal at $(x_0, f(x_0))$ is $y - y_0 = \frac{-1}{m} (x - x_0)$.

10. Testing for maxima / Minima or finding Maximum, Minimum, Extremum of a real valued function defined in a domain: Let f be a function defined on an open interval $I = (a, b)$ and is continuous at the critical point $c \in I$, then

a. If $f'(x)$ changes sign from positive to negative as x increases through c , at every point in the neighborhood of c : (i) $f'(x) > 0$ to $f'(x) < 0$ then $(c, f(c))$ is a point of **local Maxima**. (ii) $f'(x) < 0$ to $f'(x) > 0$ then $(c, f(c))$ is a point of **local Minima**.

b. If $f'(x)$ does not change sign from positive to negative as x increases through c , at every point in the neighborhood of c , then $(c, f(c))$ is called **point of inflexion**.

11. The above is a first order derivative test. The second order derivative test will establish all points of Maxima, Minima, point of inflexion as shown below:

a. $x = c$ is a point of local maxima if $f'(c) = 0$ and $f''(x) < 0$, $f(c) =$ local maximum.

b. $x = c$ is a point of local minima if $f'(c) = 0$ and $f''(x) > 0$, $f(c) =$ local minimum.

c. The test fails if $f'(c) = 0$ and $f''(x) = 0$.

d. if $f'(c) = 0$ and $f''(x) = 0$, only the first derivative test will assure point of maxima/minima.

e. if $f'(c) = 0$ and $f''(x) = 0$, and if $f'''(x) \neq 0$, then $(c, f(c))$ is **a point of inflexion**.

12. Absolute Maxima / Absolute Minima:

a. Find all points where $f'(x) = 0$ or not differentiable in the given domain.

b. Find the values of f at end points, critical points. Arrange them in ascending order. The least value is **absolute minimum** and highest value is **Absolute Maximum**.

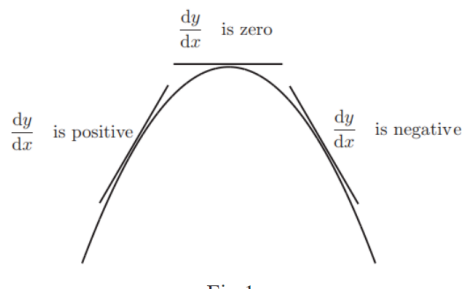
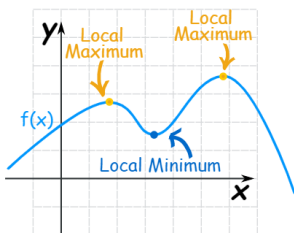
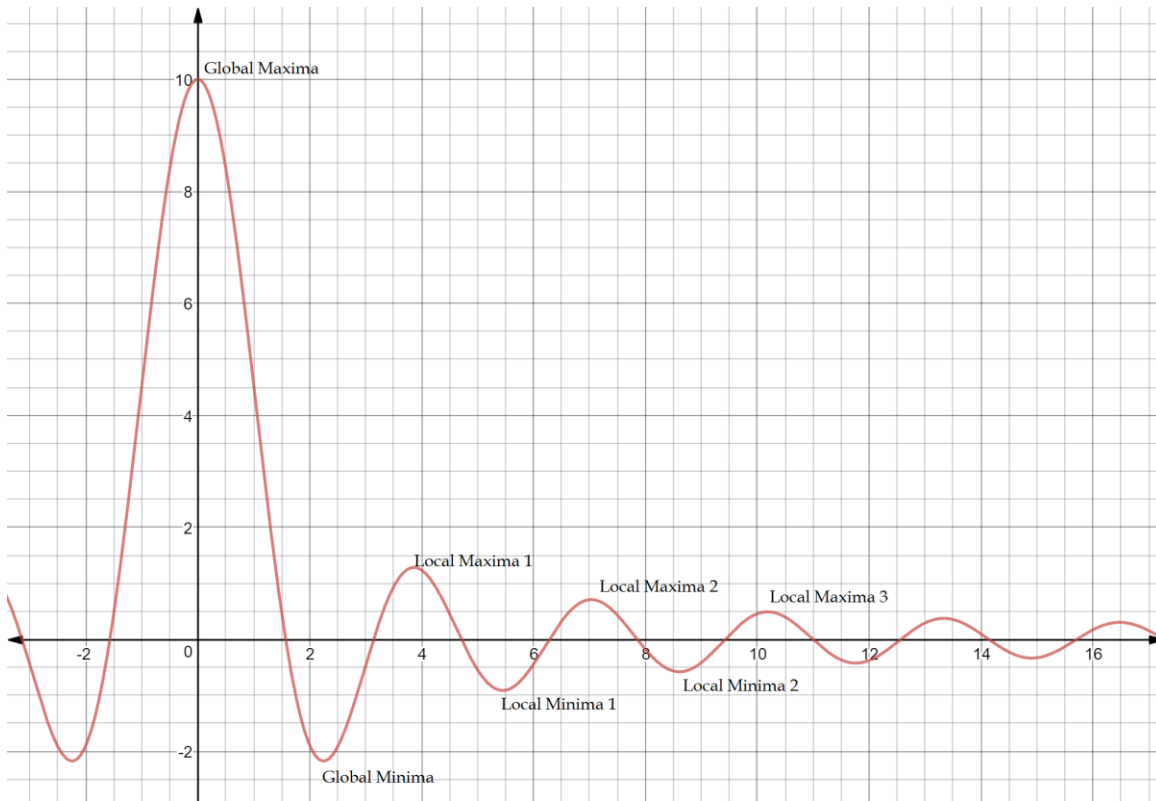
13. Global Maxima / Global Minima:

A point is known as a Global Maxima of a function when there is no other point in the domain of the function for which the value of the function is more than the value of the global maxima. Types of Global Maxima:

- Global maxima may satisfy all the conditions of local maxima. You can also understand it as the Local Maxima with the maximum value in this case.

- Alternately, the global maxima for an increasing function could be the endpoint in its domain; as it would obviously have the maximum value. In this case, it isn't a local maximum for the function.

Similarly, the *local and the global minima* can be defined. Look at the graph below to identify the different types of maxima and minima.



Problems and solved answers:

(MCQs) focuses on “Application of Derivative”.

1. What is the slope of the tangent to the curve $y = 2x/(x^2 + 1)$ at $(0, 0)$? a) 0 b) 1 c) 2 d) 3
2. The value of $f'(x)$ is -1 at the point P on a continuous curve $y = f(x)$. What is the angle which the tangent to the curve at P makes with the positive direction of x axis? a) $\pi/2$ b) $\pi/4$ c) $3\pi/4$ d) $3\pi/2$
3. What will be the differential function of $\log(x^2 + 4)$? a) $2x/(x^2 + 4) dx$ b) $2x/(x^2 - 4) dx$ c) $-2x/(x^2 + 4) dx$ d) $-2x/(x^2 - 4) dx$
4. What will be the average rate of change of the function $[y = 16 - x^2]$ between $x = 3$ and $x = 4$? a) 7 b) -7 c) 9 d) -9
5. What will be the average rate of change of the function $[y = 16 - x^2]$ at $x = 4$? a) -8 b) 8 c) -9 d) Depends on the value of x
6. What will be the value of the co-ordinate whose position of a particle moving along the parabola $y^2 = 4x$ at which the rate at of increase of the abscissa is twice the rate of increase of the ordinate? a) (1, 1) b) (2, 2) c) (3, 3) d) (4, 4)
7. A particle moving in a straight line covers a distance of x cm in t second, where $x = t^3 + 6t^2 - 15t + 18$. What will be the velocity of the particle at the end of 2 seconds? a) 20cm/sec b) 22cm/sec c) 21cm/sec d) 23cm/sec
8. A particle moving in a straight line covers a distance of x cm in t second, where $x = t^3 + 6t^2 - 15t + 18$. What will be the acceleration of the particle at the end of 2 seconds? a) 22cm/sec² b) 23cm/sec² c) 24cm/sec² d) 25cm/sec²

9. A particle moving in a straight line covers a distance of x cm in t second, where $x = t^3 + 6t^2 - 15t + 18$. When does the particle stop? a) $1/4$ second b) $1/3$ second c) 1 second d) $1/2$ second

Answers: 1. C 2. C 3. A 4. B 5. A 6. D 7. C 8. C 9. C

MCQ on Chapter -6: Application of derivatives:

1. The slope of tangent to the curve $x = t^2 + 3t - 8$, $y = 2t^2 - 2t - 5$ at the point $(2, -1)$ is

- (a) 227
- (b) 67
- (c) -67
- (d) -6

2. The interval on which the function $f(x) = 2x^3 + 9x^2 + 12x - 1$ is decreasing is

- (a) $[-1, \infty]$
- (b) $[-2, -1]$
- (c) $[-\infty, -2]$
- (d) $[-1, 1]$

3. Let the $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = 2x + \cos x$, then f

- (a) has a minimum at $x = 3t$
- (b) has a maximum, at $x = 0$
- (c) is a decreasing function
- (d) is an increasing function

4. $y = x(x - 3)^2$ decreases for the values of x given by

- (a) $1 < x < 3$
- (b) $x < 0$
- (c) $x > 0$
- (d) $0 < x < 32$

5. The function $f(x) = 4 \sin^3 x - 6 \sin^2 x + 12 \sin x + 100$ is strictly

- (a) increasing in $(\pi, 3\pi/2)$
- (b) decreasing in $(\pi/2, \pi)$
- (c) decreasing in $[-\pi/2, \pi/2]$
- (d) decreasing in $[0, \pi/2]$

6. Which of the following functions is decreasing on $(0, \pi/2)$?

- (a) $\sin 2x$
- (b) $\tan x$
- (c) $\cos x$
- (d) $\cos 3x$

7. The function $f(x) = \tan x - x$

- (a) always increases
- (b) always decreases
- (c) sometimes increases and sometimes decreases
- (d) never increases

8. If x is real, the minimum value of $x^2 - 8x + 17$ is
- (a) -1
 - (b) 0
 - (c) 1
 - (d) 2
9. The smallest value of the polynomial $x^3 - 18x^2 + 96x$ in $[0, 9]$ is
- (a) 126
 - (b) 0
 - (c) 135
 - (d) 160
10. The function $f(x) = 2x^3 - 3x^2 - 12x + 4$ has
- (a) two points of local maximum
 - (b) two points of local minimum
 - (c) one maxima and one minima
 - (d) no maxima or minima
11. The maximum value of $\sin x - \cos x$ is
- (a) $1/4$
 - (b) $1/2$
 - (c) $\sqrt{2}$
 - (d) $2\sqrt{2}$
12. At $x = 5\pi/6$, $f(x) = 2 \sin 3x + 3 \cos 3x$ is
- (a) maximum
 - (b) minimum
 - (c) zero
 - (d) neither maximum nor minimum
13. Maximum slope of the curve $y = -x^3 + 3x^2 + 9x - 27$ is
- (a) 0
 - (b) 12
 - (c) 16
 - (d) 32
14. $f(x) = x^x$ has a stationary point at
- (a) $x = e$
 - (b) $x = 1/e$
 - (c) $x = 1$
 - (d) $x = \sqrt{e}$
15. The maximum value of $(1/x)^x$ is
- (a) e
 - (b) e^2
 - (c) $e^{1/x}$
 - (d) $(1/e)^{1/e}$

16. If the volume of a sphere is increasing at a constant rate, then the rate at which its radius is increasing is
- a constant
 - proportional to the radius
 - inversely proportional to the radius
 - inversely proportional to the surface area
17. A particle is moving along the curve $x = at^2 + bt + c$. If $ac = b^2$, then particle would be moving with uniform
- rotation
 - velocity
 - acceleration
 - retardation
18. The distance y metres covered by a body in t seconds, is given by $s = 3t^2 - 8t + 5$. The body will stop after
- 1 s
 - $3/4$ s
 - $4/3$ s
 - 4 s
19. The position of a point in time y is given by $x = a + bt + ct^2$, $y = at + bt^2$. Its acceleration at time y is
- $b - c$
 - $b + c$
 - $2b - 2c$
 - $2\sqrt{b^2 + c^2}$
20. The function $f(x) = \log(1 + x) - \frac{2x}{2+x}$ is increasing on
- $(-1, \infty)$
 - $(-\infty, 0)$
 - $(-\infty, \infty)$
 - None of these
21. $f(x) = \frac{e^{2x}-1}{e^{2x}+1}$ is
- an increasing function
 - a decreasing function
 - an even function
 - None of these
22. If $f(x) = \frac{x}{\sin x}$ and $g(x) = \frac{x}{\tan x}$, $0 < x \leq 1$, then in the interval
- both $f(x)$ and $g(x)$ are increasing functions
 - both $f(x)$ and $g(x)$ are decreasing functions
 - $f(x)$ is an increasing function
 - $g(x)$ is an increasing function
23. The function $f(x) = \cot^{-1} x + x$ increases in the interval
- $(1, \infty)$
 - $(-1, \infty)$

(c) $(0, \infty)$

(d) $(-\infty, \infty)$

24. The function $f(x) = \frac{x}{\log x}$ increases on the interval

(a) $(0, \infty)$

(b) $(0, e)$

(c) (e, ∞)

(d) None of these

25. The value of b for which the function $f(x) = \sin x - bx + c$ is decreasing for $x \in \mathbb{R}$ is given by

(a) $b < 1$

(b) $b \geq 1$

(c) $b > 1$

(d) $b \leq 1$

26. If $f(x) = x^3 - 6x^2 + 9x + 3$ be a decreasing function, then x lies in

(a) $(-\infty, -1) \cap (3, \infty)$

(b) $(1, 3)$

(c) $(3, \infty)$

(d) None of these

27. The function $f(x) = 1 - x^3 - x^5$ is decreasing for

(a) $1 < x < 5$

(b) $x < 1$

(c) $x > 1$

(d) all values of x

28. Function, $f(x) = \frac{\lambda \sin x + 6 \cos x}{2 \sin x + 3 \cos x}$ is monotonic increasing, if

(a) $\lambda > 1$

(b) $\lambda < 1$

(c) $-2 < \lambda < 2$

(d) $\lambda > 4$

29. The length of the longest interval, in which the function $3 \sin x - 4 \sin^3 x$ is increasing is

(a) π^3

(b) π^2

(c) $3\pi^2$

(d) π

30. $2x^3 - 6x + 5$ is an increasing function, if

(a) $0 < x < 1$

(b) $-1 < x < 1$

(c) $x < -1$ or $x > 1$

(d) $-1 < x < -12$

31. The function $f(x) = x + \cos x$ is

(a) always increasing

(b) always decreasing

(c) increasing for certain range of x

(d) None of these

32. The function which is neither decreasing nor increasing in $(\pi/2, 3\pi/2)$ is

(a) $\operatorname{cosec} x$

(b) $\tan x$

(c) x^2

(d) $|x - 1|$

33. The interval in which the function $y = x^3 + 5x^2 - 1$ is decreasing, is

(a) $(0, 13)$

(b) $(0, 10)$

(c) $(-10/3, 0)$

(d) None of these

Answers: 1. c 2. b 3. d 4. a 5. c 6. c 7. a 8. d 9. b 10. c 11. b 12. d 13. a 14. b 15. d 16. d 17.* c 18. c 19.* d
20. a 21. a 22. c 23. d 24. c 25. b 26. b 27. d 28.* c 29. a 30. c 31. a 32. b 33. C

Answer to Problems in NCERT :

Application of derivative:

Find the maximum and minimum values, if any, of the following functions given by

$$(i) f(x) = (2x - 1)^2 + 3$$

$$f(x) = (2x - 1)^2 + 3$$



Square of number cant be negative

Hence,

$$\text{Minimum value of } (2x - 1)^2 = 0$$

$$\text{Minimum value of } (2x - 1)^2 + 3 = 0 + 3 = 3$$

Also,

there is no maximum value of x

\therefore There is **no maximum value** of $f(x)$

2. Find the maximum and minimum values, if any, of the following functions given by

$$(iii) f(x) = -(x - 1)^2 + 10$$

$$f(x) = -(x - 1)^2 + 10$$

Finding $f'(x)$

Diff w.r.t x

$$f'(x) = \frac{d(-(x-1)^2+10)}{dx}$$

Putting $f'(x) = 0$

$$f'(x) = -2(x - 1) \left(\frac{d(x-1)}{dx} \right) + 0$$

$$-2(x - 1) = 0$$

$$f'(x) = -2(x - 1)(1 - 0) + 0$$

$$(x - 1) = 0$$

$$f'(x) = -2(x - 1)$$

$$x = 1$$

Value of x	Sign of $f'(x) = 2(x - 1)$	Maxima or minima
At $x = 1$ <div style="display: inline-block; vertical-align: middle; margin-left: 10px;"> $\left\{ \begin{array}{l} \text{If } x < 1 \text{ (say 0.9)} \\ \text{If } x > 1 \text{ (say 1.1)} \end{array} \right.$ </div>	$(+)$ $(-)$	Since sign of $f'(x)$ changes from positive to negative, it is Maxima

Hence, $x = 1$ is point of Maxima

& No point of Minima

Thus,

$f(x)$ has maximum value at $x = 1$

Putting $x = 1$ in $f(x)$

$$f(x) = -(x - 1)^2 + 10$$

$$f(1) = -(1 - 1)^2 + 10$$

$$= 0 + 10$$

$$= 10$$

Maximum value of $f(x)$ is 10

There is no Minimum

Find the maximum and minimum values, if any, of the following functions given by

(ii) $f(x) = -|x + 1| + 3$

$$f(x) = -|x + 1| + 3$$

We know that $|x + 1| \geq 0$

So, $-|x + 1| \leq 0$

Maximum value of $g(x)$

$$= \text{maximum value of } -|x + 1| + 3$$

$$= 0 + 3$$

$$= 3$$

Hence **maximum value** of $f(x)$ is **3**

And there is **no minimum value** of $f(x)$

4 Find the maximum and minimum values, if any, of the following functions given by

(iii) $h(x) = \sin(2x) + 5$

$$h(x) = \sin(2x) + 5$$

We know that

$$-1 \leq \sin \theta \leq 1$$

$$-1 \leq \sin 2x \leq 1$$

Adding 5 both sides

$$-1 + 5 \leq \sin 2x + 5 \leq 1 + 5$$

$$4 \leq \sin 2x + 5 \leq 6$$

$$4 \leq f(x) \leq 6$$

Hence **Maximum value of $f(x) = 6$**

& **Minimum value of $f(x) = 4$**

5 Find the maximum and minimum values, if any, of the following functions given by

(iv) $f(x) = |\sin 4x + 3|$

$$f(x) = |\sin 4x + 3|$$

We know that

$$-1 \leq \sin \theta \leq 1$$

So, $-1 \leq \sin 4x \leq 1$

Adding 3 both sides

$$-1 + 3 \leq \sin 4x + 3 \leq 1 + 3$$

$$2 \leq \sin 4x + 3 \leq 4$$

Taking modulus

$$|2| \leq |\sin 4x + 3| \leq |4|$$

$$2 \leq |\sin 4x + 3| \leq |4|$$

$$2 \leq f(x) \leq 4$$

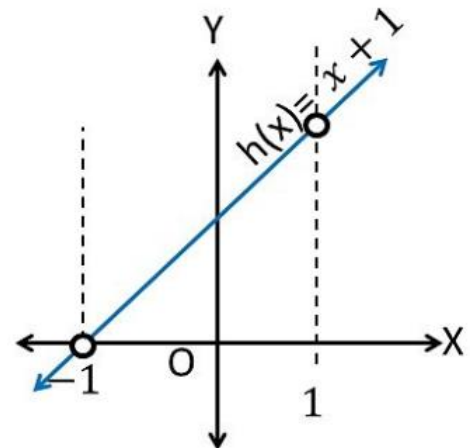
Hence **Maximum value of $f(x)$ is 4**

& **Minimum value of $f(x)$ is 2**

7 Find the maximum and minimum values, if any, of the following functions given by

$$(v) h(x) = x + 1, \quad x \in (-1, 1)$$

Drawing graph of $f(x) = x + 1$



$h(x)$ have Maximum value of point closest to $x = 1$

& Minimum value of point closest to $x = -1$

$h(x)$ have Maximum value of point closest to $x = 1$

& Minimum value of point closest to $x = -1$

but its not possible to locate such points

Thus the given **function has neither the maximum value nor minimum value**

8 Find the local maxima and local minima, if any, of the following functions. Find also the local maximum and the local minimum values, as the case may be:

$$(iii) h(x) = \sin x + \cos x, \quad 0 < x < \frac{\pi}{2}$$

$$\text{Putting } h'(x) = 0$$

$$\cos x - \sin x = 0$$

$$\cos x = \sin x$$

$$h(x) = \sin x + \cos x, \quad 0 < x < \frac{\pi}{2}$$

$$1 = \frac{\sin x}{\cos x}$$

Finding $h'(x)$

$$1 = \tan x$$

$$h'(x) = \frac{d(\sin x + \cos x)}{dx}$$

$$\tan x = 1$$

$$h'(x) = \cos x - \sin x$$

$$\therefore x = 45^\circ = \frac{\pi}{4}$$

$$\text{Putting } x = \frac{\pi}{4}$$

$$h''\left(\frac{\pi}{4}\right) = -\sin\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{4}\right)$$

Finding $h''(x)$

$$= -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}$$

$$h'(x) = \cos x - \sin x$$

$$= \frac{-2}{\sqrt{2}}$$

$$h''(x) = -\sin x - \cos x$$

$$= -\sqrt{2}$$

f has Maximum value at $x = \frac{\pi}{4}$

$$f(x) = \sin x + \cos x$$

$$f\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right)$$

Since $h''(x) < 0$ when $x = \frac{\pi}{4}$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

$\therefore x = \frac{\pi}{4}$ is point of Local Maxima

- 9 Find the local maxima and local minima, if any, of the following functions. Find also the local maximum and the local minimum values, as the case may be:

$$(iv) f(x) = \sin x - \cos x, \quad 0 < x < 2\pi$$

$$f(x) = \sin x - \cos x, \quad 0 < x < 2\pi$$

Finding $f'(x)$

$$f'(x) = \cos x - (-\sin x)$$

$$f'(x) = \cos x + \sin x$$

$$1 = \frac{-\sin x}{\cos x}$$

$$\frac{-\sin x}{\cos x} = 1$$

$$-\tan x = 1$$

$$\tan x = -1$$

Putting $f'(x) = 0$

$$\cos x + \sin x = 0$$

$$\cos x = -\sin x$$

Since $0 < x < 2\pi$ & $\tan x$ is negative

$\tan \theta$ lies in either 2nd or 4th quadrant

So, value of x is

$$x = \frac{3\pi}{4} \text{ OR } \frac{7\pi}{4}$$

Q7 : Find both the maximum value and the minimum value of $3x^4 - 8x^3 + 12x^2 - 48x + 25$ on the interval $[0, 3]$

Answer :

Let $f(x) = 3x^4 - 8x^3 + 12x^2 - 48x + 25$.

$$\begin{aligned}\therefore f'(x) &= 12x^3 - 24x^2 + 24x - 48 \\ &= 12(x^3 - 2x^2 + 2x - 4) \\ &= 12[x^2(x-2) + 2(x-2)] \\ &= 12(x-2)(x^2 + 2)\end{aligned}$$

Now, $f'(x) = 0$ gives $x = 2$ or $x^2 + 2 = 0$ for which there are no real roots.

Therefore, we consider only $x = 2 \in [0, 3]$.

Now, we evaluate the value of f at critical point $x = 2$ and at the end points of the interval $[0, 3]$.

$$f(2) = 3(16) - 8(8) + 12(4) - 48(2) + 25$$

$$\begin{aligned}f(2) &= 3(16) - 8(8) + 12(4) - 48(2) + 25 \\ &= 48 - 64 + 48 - 96 + 25 \\ &= -39\end{aligned}$$

$$\begin{aligned}f(0) &= 3(0) - 8(0) + 12(0) - 48(0) + 25 \\ &= 25\end{aligned}$$

$$\begin{aligned}f(3) &= 3(81) - 8(27) + 12(9) - 48(3) + 25 \\ &= 243 - 216 + 108 - 144 + 25 = 16\end{aligned}$$

Hence, we can conclude that the absolute maximum value of f on $[0, 3]$ is 25 occurring at $x = 0$ and the absolute minimum value of f at $[0, 3]$ is - 39 occurring at $x = 2$.

Q8 : At what points in the interval $[0, 2\pi]$, does the function $\sin 2x$ attain its maximum value?

Answer :

$$\text{Let } f(x) = \sin 2x.$$

$$\therefore f'(x) = 2 \cos 2x$$

Now,

$$f'(x) = 0 \Rightarrow \cos 2x = 0$$

$$\Rightarrow 2x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$$

$$\Rightarrow x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

Then, we evaluate the values of f at critical points $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ and at the end points of the interval $[0, 2\pi]$.

$$f\left(\frac{\pi}{4}\right) = \sin \frac{\pi}{2} = 1, f\left(\frac{3\pi}{4}\right) = \sin \frac{3\pi}{2} = -1$$

Then, we evaluate the values of f at critical points $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ and at the end points of the interval $[0, 2\pi]$.

$$f\left(\frac{\pi}{4}\right) = \sin \frac{\pi}{2} = 1, f\left(\frac{3\pi}{4}\right) = \sin \frac{3\pi}{2} = -1$$

$$f\left(\frac{5\pi}{4}\right) = \sin \frac{5\pi}{2} = 1, f\left(\frac{7\pi}{4}\right) = \sin \frac{7\pi}{2} = -1$$

$$f(0) = \sin 0 = 0, f(2\pi) = \sin 2\pi = 0$$

Hence, we can conclude that the absolute maximum value of f on $[0, 2\pi]$ is occurring at $x = \frac{\pi}{4}$ and

$$x = \frac{5\pi}{4}.$$

Q9 : What is the maximum value of the function $\sin x + \cos x$?

Answer :

Let $f(x) = \sin x + \cos x$.

$$\therefore f'(x) = \cos x - \sin x$$

$$f'(x) = 0 \Rightarrow \sin x = \cos x \Rightarrow \tan x = 1 \Rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4}, \dots$$

$$f''(x) = -\sin x - \cos x = -(\sin x + \cos x)$$

Now, $f''(x)$ will be negative when $(\sin x + \cos x)$ is positive i.e., when $\sin x$ and $\cos x$ are both positive. Also, we know that $\sin x$ and $\cos x$ both are positive in the first quadrant. Then, $f''(x)$ will be negative when $x \in \left(0, \frac{\pi}{2}\right)$.

Thus, we consider $x = \frac{\pi}{4}$.

$$f''\left(\frac{\pi}{4}\right) = -\left(\sin \frac{\pi}{4} + \cos \frac{\pi}{4}\right) = -\left(\frac{2}{\sqrt{2}}\right) = -\sqrt{2} < 0$$

\therefore By second derivative test, f will be the maximum at $x = \frac{\pi}{4}$ and the maximum value of f is

$$f\left(\frac{\pi}{4}\right) = \sin \frac{\pi}{4} + \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}.$$

Q10 : Find the maximum value of $2x^3 - 24x + 107$ in the interval $[1, 3]$. Find the maximum value of the same function in $[-3, -1]$.

Answer :

Let $f(x) = 2x^3 - 24x + 107$.

$$\therefore f'(x) = 6x^2 - 24 = 6(x^2 - 4)$$

Now,

$$f'(x) = 0 \Rightarrow 6(x^2 - 4) = 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$$

We first consider the interval $[1, 3]$.

Then, we evaluate the value of f at the critical point $x = 2 \in [1, 3]$ and at the end points of the interval $[1, 3]$.

$$f(2) = 2(8) - 24(2) + 107 = 16 - 48 + 107 = 75$$

$$f(1) = 2(1) - 24(1) + 107 = 2 - 24 + 107 = 85$$

$$f(3) = 2(27) - 24(3) + 107 = 54 - 72 + 107 = 89$$

Hence, the absolute maximum value of $f(x)$ in the interval $[1, 3]$ is 89 occurring at $x = 3$.

Next, we consider the interval $[-3, -1]$.

Evaluate the value of f at the critical point $x = -2 \in [-3, -1]$ and at the end points of the interval $[-3, -1]$.

$$f(-3) = 2(-27) - 24(-3) + 107 = -54 + 72 + 107 = 125$$

$$f(-1) = 2(-1) - 24(-1) + 107 = -2 + 24 + 107 = 129$$

$$f(-2) = 2(-8) - 24(-2) + 107 = -16 + 48 + 107 = 139$$

Hence, the absolute maximum value of $f(x)$ in the interval $[-3, -1]$ is 139 occurring at $x = -2$.

Q12 : Find the maximum and minimum values of $x + \sin 2x$ on $[0, 2\pi]$.

Answer :

Let $f(x) = x + \sin 2x$.

$$\therefore f'(x) = 1 + 2\cos 2x$$

$$\text{Now, } f'(x) = 0 \Rightarrow \cos 2x = -\frac{1}{2} = -\cos \frac{\pi}{3} = \cos \left(\pi - \frac{\pi}{3} \right) = \cos \frac{2\pi}{3}$$

$$2x = 2n\pi \pm \frac{2\pi}{3}, \quad n \in \mathbb{Z}$$

$$\Rightarrow x = n\pi \pm \frac{\pi}{3}, \quad n \in \mathbb{Z}$$

$$\Rightarrow x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3} \in [0, 2\pi]$$

Then, we evaluate the value of f at critical points $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$ and at the end points of the interval $[0, 2\pi]$.

interval $[0, 2\pi]$.

$$f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} + \sin \frac{2\pi}{3} = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$$

$$f\left(\frac{2\pi}{3}\right) = \frac{2\pi}{3} + \sin \frac{4\pi}{3} = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$$

$$f\left(\frac{4\pi}{3}\right) = \frac{4\pi}{3} + \sin \frac{8\pi}{3} = \frac{4\pi}{3} + \frac{\sqrt{3}}{2}$$

$$f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{3} + \sin \frac{10\pi}{3} = \frac{5\pi}{3} - \frac{\sqrt{3}}{2}$$

$$f(0) = 0 + \sin 0 = 0$$

$$f(2\pi) = 2\pi + \sin 4\pi = 2\pi + 0 = 2\pi$$

Hence, we can conclude that the absolute maximum value of $f(x)$ in the interval $[0, 2\pi]$ is 2π occurring at $x = 2\pi$ and the absolute minimum value of $f(x)$ in the interval $[0, 2\pi]$ is 0 occurring at $x = 0$.

Q15 : Find two positive numbers x and y such that their sum is 35 and the product x^2y^5 is a maximum

Answer :

Let one number be x . Then, the other number is $y = (35 - x)$.

Let $P(x) = x^2y^5$. Then, we have:

$$\begin{aligned}
 P(x) &= x^2(35-x)^5 & \text{And, } P'(x) &= 7(35-x)^4(10-x) + 7x[-(35-x)^4 - 4(35-x)^3(10-x)] \\
 \therefore P'(x) &= 2x(35-x)^5 - 5x^2(35-x)^4 & &= 7(35-x)^4(10-x) - 7x(35-x)^4 - 28x(35-x)^3(10-x) \\
 &= x(35-x)^4[2(35-x) - 5x] & &= 7(35-x)^3[(35-x)(10-x) - x(35-x) - 4x(10-x)] \\
 &= x(35-x)^4(70-7x) & &= 7(35-x)^3[350-45x+x^2-35x+x^2-40x+4x^2] \\
 &= 7x(35-x)^4(10-x) & &= 7(35-x)^3(6x^2-120x+350)
 \end{aligned}$$

Now, $P'(x) = 0 \Rightarrow x = 0, x = 35, x = 10$

When $x = 35, f'(x) = f(x) = 0$ and $y = 35 - 35 = 0$. This will make the product x^2y^5 equal to 0.

When $x = 0, y = 35 - 0 = 35$ and the product x^2y^5 will be 0.

$\therefore x = 0$ and $x = 35$ cannot be the possible values of x .

When $x = 10$, we have:

$$\begin{aligned}
 P''(x) &= 7(35-10)^3(6 \times 100 - 120 \times 10 + 350) \\
 &= 7(25)^3(-250) < 0
 \end{aligned}$$

\therefore By second derivative test, $P(x)$ will be the maximum when $x = 10$ and $y = 35 - 10 = 25$.

Hence, the required numbers are 10 and 25.

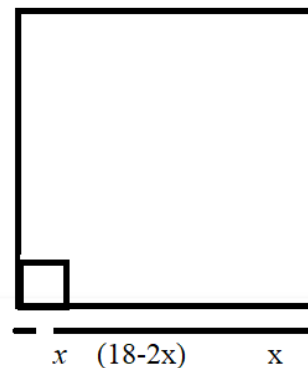
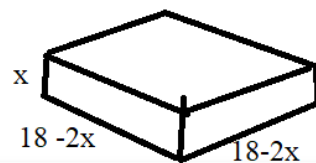
Q17 : A square piece of tin of side 18 cm is to be made into a box without top, by cutting a square from each corner and folding up the flaps to form the box. What should be the side of the square to be cut off so that the volume of the box is the maximum possible?

Answer :

Let the side of the square to be cut off be x cm. Then, the length and the breadth of the box will be $(18 - 2x)$ cm each and the height of the box is x cm.

Therefore, the volume $V(x)$ of the box is given by,

$$\begin{aligned}
 V(x) &= x(18 - 2x)^2 \\
 \therefore V'(x) &= (18 - 2x)^2 - 4x(18 - 2x) \\
 &= (18 - 2x)[18 - 2x - 4x] \\
 &= (18 - 2x)(18 - 6x) \\
 &= 6 \times 2(9 - x)(3 - x) \\
 &= 12(9 - x)(3 - x)
 \end{aligned}$$



$$\begin{aligned}
 \text{And, } V''(x) &= 12[-(9-x)-(3-x)] \\
 &= -12(9-x+3-x) \\
 &= -12(12-2x) \\
 &= -24(6-x)
 \end{aligned}$$

$$\text{Now, } V'(x) = 0 \Rightarrow x = 9 \text{ or } x = 3$$

If $x = 9$, then the length and the breadth will become 0.

$$\therefore x \neq 9.$$

$$\Rightarrow x = 3.$$

$$\text{Now, } V''(3) = -24(6-3) = -72 < 0$$

\therefore By second derivative test, $x = 3$ is the point of maxima of V .

Hence, if we remove a square of side 3 cm from each corner of the square tin and make a box from the remaining sheet, then the volume of the box obtained is the largest possible.

Q18 : A rectangular sheet of tin 45 cm by 24 cm is to be made into a box without top, by cutting off square from each corner and folding up the flaps. What should be the side of the square to be cut off so that the volume of the box is the maximum possible?

Answer :

Let the side of the square to be cut off be x cm. Then, the height of the box is x , the length is $45 - 2x$, and the breadth is $24 - 2x$.

Therefore, the volume $V(x)$ of the box is given by,

$$\begin{aligned}
 V(x) &= x(45-2x)(24-2x) \\
 &= x(1080-90x-48x+4x^2) \\
 &= 4x^3-138x^2+1080x \\
 \therefore V'(x) &= 12x^2-276x+1080 \\
 &= 12(x^2-23x+90) \\
 &= 12(x-18)(x-5)
 \end{aligned}$$

$$V''(x) = 24x - 276 = 12(2x - 23)$$

$$\text{Now, } V'(x) = 0 \Rightarrow x = 18 \text{ and } x = 5$$

It is not possible to cut off a square of side 18 cm from each corner of the rectangular sheet. Thus, x cannot be equal to 18.

$$\therefore x = 5$$

$$\text{Now, } V''(5) = 12(10-23) = 12(-13) = -156 < 0$$

\therefore By second derivative test, $x = 5$ is the point of maxima.

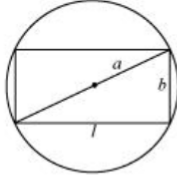
Hence, the side of the square to be cut off to make the volume of the box maximum possible is 5 cm.

Q19 : Show that of all the rectangles inscribed in a given fixed circle, the square has the maximum area.

Answer :

Let a rectangle of length l and breadth b be inscribed in the given circle of radius a .

Then, the diagonal passes through the centre and is of length $2a$ cm.



Now, by applying the Pythagoras theorem, we have:

$$(2a)^2 = l^2 + b^2$$

$$\Rightarrow b^2 = 4a^2 - l^2$$

$$\Rightarrow b = \sqrt{4a^2 - l^2}$$

$$\therefore \text{Area of the rectangle, } A = l\sqrt{4a^2 - l^2}$$

$$\therefore \frac{dA}{dl} = \sqrt{4a^2 - l^2} + l \frac{1}{2\sqrt{4a^2 - l^2}} (-2l) = \sqrt{4a^2 - l^2} - \frac{l^2}{\sqrt{4a^2 - l^2}}$$

$$= \frac{4a^2 - 2l^2}{\sqrt{4a^2 - l^2}}$$

$$\frac{d^2 A}{dl^2} = \frac{\sqrt{4a^2 - l^2} (-4l) - (4a^2 - 2l^2) \frac{(-2l)}{2\sqrt{4a^2 - l^2}}}{(4a^2 - l^2)}$$

$$= \frac{(4a^2 - l^2)(-4l) + l(4a^2 - 2l^2)}{(4a^2 - l^2)^{\frac{3}{2}}}$$

$$= \frac{-12a^2 l + 2l^3}{(4a^2 - l^2)^{\frac{3}{2}}} = \frac{-2l(6a^2 - l^2)}{(4a^2 - l^2)^{\frac{3}{2}}}$$

$$\text{Now, } \frac{dA}{dl} = 0 \text{ gives } 4a^2 = 2l^2 \Rightarrow l = \sqrt{2}a$$

$$\Rightarrow b = \sqrt{4a^2 - 2a^2} = \sqrt{2a^2} = \sqrt{2}a$$

Now, when $l = \sqrt{2}a$,

$$\frac{d^2 A}{dl^2} = \frac{-2(\sqrt{2}a)(6a^2 - 2a^2)}{2\sqrt{2}a^3} = \frac{-8\sqrt{2}a^3}{2\sqrt{2}a^3} = -4 < 0$$

\therefore By the second derivative test, when $l = \sqrt{2}a$, then the area of the rectangle is the maximum.

Since $l = b = \sqrt{2}a$, the rectangle is a square.

Hence, it has been proved that of all the rectangles inscribed in the given fixed circle, the square has the maximum area.

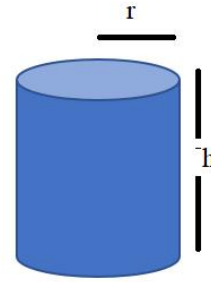
Q20 : Show that the right circular cylinder of given surface and maximum volume is such that its height is equal to the diameter of the base.

Answer :

Let r and h be the radius and height of the cylinder respectively.

Then, the surface area (S) of the cylinder is given by,

$$\begin{aligned} S &= 2\pi r^2 + 2\pi r h \\ \Rightarrow h &= \frac{S - 2\pi r^2}{2\pi r} \\ &= \frac{S}{2\pi} \left(\frac{1}{r} \right) - r \end{aligned}$$



Let V be the volume of the cylinder. Then,

$$V = \pi r^2 h = \pi r^2 \left[\frac{S}{2\pi} \left(\frac{1}{r} \right) - r \right] = \frac{Sr}{2} - \pi r^3$$

$$\text{Then, } \frac{dV}{dr} = \frac{S}{2} - 3\pi r^2, \quad \frac{d^2V}{dr^2} = -6\pi r$$

$$\text{Now, } \frac{dV}{dr} = 0 \Rightarrow \frac{S}{2} = 3\pi r^2 \Rightarrow r^2 = \frac{S}{6\pi}$$

$$\text{When } r^2 = \frac{S}{6\pi}, \text{ then } \frac{d^2V}{dr^2} = -6\pi \left(\sqrt{\frac{S}{6\pi}} \right) < 0.$$

\therefore By second derivative test, the volume is the maximum when $r^2 = \frac{S}{6\pi}$.

$$\text{Now, when } r^2 = \frac{S}{6\pi}, \text{ then } h = \frac{6\pi r^2}{2\pi} \left(\frac{1}{r} \right) - r = 3r - r = 2r.$$

Hence, the volume is the maximum when the height is twice the radius i.e., when the height is equal to the diameter.

Q21 : Of all the closed cylindrical cans (right circular), of a given volume of 100 cubic centimetres, find the dimensions of the can which has the minimum surface area?

Answer :

Let r and h be the radius and height of the cylinder respectively.

Then, volume (V) of the cylinder is given by,

$$V = \pi r^2 h = 100 \quad (\text{given})$$

$$\therefore h = \frac{100}{\pi r^2}$$

Surface area (S) of the cylinder is given by,

$$S = 2\pi r^2 + 2\pi r h = 2\pi r^2 + \frac{200}{r}$$

$$\therefore \frac{dS}{dr} = 4\pi r - \frac{200}{r^2}, \quad \frac{d^2S}{dr^2} = 4\pi + \frac{400}{r^3}$$

$$\frac{dS}{dr} = 0 \Rightarrow 4\pi r = \frac{200}{r^2}$$

$$\Rightarrow r^3 = \frac{200}{4\pi} = \frac{50}{\pi}$$

$$\Rightarrow r = \left(\frac{50}{\pi}\right)^{\frac{1}{3}}$$

Now, it is observed that when $r = \left(\frac{50}{\pi}\right)^{\frac{1}{3}}$, $\frac{d^2S}{dr^2} > 0$.

\therefore By second derivative test, the surface area is the minimum when the radius of the cylinder is

$$\left(\frac{50}{\pi}\right)^{\frac{1}{3}} \text{ cm.}$$

$$\text{When } r = \left(\frac{50}{\pi}\right)^{\frac{1}{3}}, \quad h = \frac{100}{\pi \left(\frac{50}{\pi}\right)^{\frac{2}{3}}} = \frac{2 \times 50}{(50)^{\frac{2}{3}} (\pi)^{1-\frac{2}{3}}} = 2 \left(\frac{50}{\pi}\right)^{\frac{1}{3}} \text{ cm.}$$

Hence, the required dimensions of the can which has the minimum surface area is given by radius

$$= \left(\frac{50}{\pi}\right)^{\frac{1}{3}} \text{ cm and height} = 2 \left(\frac{50}{\pi}\right)^{\frac{1}{3}} \text{ cm.}$$

Q22 : A wire of length 28 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. What should be the length of the two pieces so that the combined area of the square and the circle is minimum?

Answer :

Let a piece of length l be cut from the given wire to make a square.

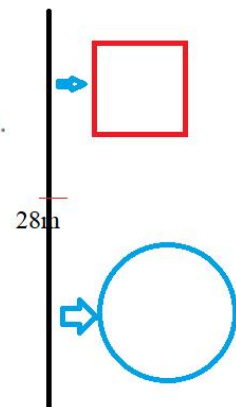
Then, the other piece of wire to be made into a circle is of length $(28 - l)$ m.

Now, side of square = $\frac{l}{4}$.

Let r be the radius of the circle. Then, $2\pi r = 28 - l \Rightarrow r = \frac{1}{2\pi}(28 - l)$.

The combined areas of the square and the circle (A) is given by,

$$A = (\text{side of the square})^2 + \pi r^2$$



$$\begin{aligned}
&= \frac{l^2}{16} + \pi \left[\frac{1}{2\pi} (28-l) \right]^2 \\
&= \frac{l^2}{16} + \frac{1}{4\pi} (28-l)^2 \\
\therefore \frac{dA}{dl} &= \frac{2l}{16} + \frac{2}{4\pi} (28-l)(-1) = \frac{l}{8} - \frac{1}{2\pi} (28-l) \\
\frac{d^2A}{dl^2} &= \frac{1}{8} + \frac{1}{2\pi} > 0 \\
\text{Now, } \frac{dA}{dl} = 0 &\Rightarrow \frac{l}{8} - \frac{1}{2\pi} (28-l) = 0 \\
&\Rightarrow \frac{\pi l - 4(28-l)}{8\pi} = 0 \\
&\Rightarrow (\pi+4)l - 112 = 0 \\
&\Rightarrow l = \frac{112}{\pi+4}
\end{aligned}$$

Thus, when $l = \frac{112}{\pi+4}$, $\frac{d^2A}{dl^2} > 0$.

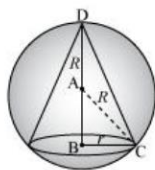
\therefore By second derivative test, the area (A) is the minimum when $l = \frac{112}{\pi+4}$.

Hence, the combined area is the minimum when the length of the wire in making the square is $\frac{112}{\pi+4}$ cm while the length of the wire in making the circle is $28 - \frac{112}{\pi+4} = \frac{28\pi}{\pi+4}$ cm.

Q23 : Prove that the volume of the largest cone that can be inscribed in a sphere of radius R is $\frac{8}{27}$ of the volume of the sphere.

Answer :

Let r and h be the radius and height of the cone respectively inscribed in a sphere of radius R .



Let V be the volume of the cone.

Then, $V = \frac{1}{3} \pi r^2 h$

Height of the cone is given by,

$$h = R + AB = R + \sqrt{R^2 - r^2} \quad [\text{ABC is a right triangle}]$$

$$\begin{aligned} \therefore V &= \frac{1}{3} \pi r^2 \left(R + \sqrt{R^2 - r^2} \right) \\ &= \frac{1}{3} \pi r^2 R + \frac{1}{3} \pi r^2 \sqrt{R^2 - r^2} \\ \therefore \frac{dV}{dr} &= \frac{2}{3} \pi r R + \frac{2}{3} \pi r \sqrt{R^2 - r^2} + \frac{1}{3} \pi r^2 \cdot \frac{(-2r)}{2\sqrt{R^2 - r^2}} \\ &= \frac{2}{3} \pi r R + \frac{2}{3} \pi r \sqrt{R^2 - r^2} - \frac{1}{3} \pi \frac{r^3}{\sqrt{R^2 - r^2}} \\ &= \frac{2}{3} \pi r R + \frac{2\pi r (R^2 - r^2) - \pi r^3}{3\sqrt{R^2 - r^2}} \\ &= \frac{2}{3} \pi r R + \frac{2\pi r R^2 - 3\pi r^3}{3\sqrt{R^2 - r^2}} \end{aligned}$$

$$\begin{aligned} \frac{d^2V}{dr^2} &= \frac{2\pi R}{3} + \frac{3\sqrt{R^2 - r^2} (2\pi R^2 - 9\pi r^2) - (2\pi r R^2 - 3\pi r^3) \cdot \frac{(-2r)}{6\sqrt{R^2 - r^2}}}{9(R^2 - r^2)} \\ &= \frac{2}{3} \pi R + \frac{9(R^2 - r^2)(2\pi R^2 - 9\pi r^2) + 2\pi r^2 R^2 + 3\pi r^4}{27(R^2 - r^2)^{\frac{3}{2}}} \end{aligned}$$

$$\begin{aligned} \text{Now, } \frac{dV}{dr} = 0 &\Rightarrow \frac{2}{3} \pi r R = \frac{3\pi r^3 - 2\pi r R^2}{3\sqrt{R^2 - r^2}} \\ \Rightarrow 2R &= \frac{3r^2 - 2R^2}{\sqrt{R^2 - r^2}} \Rightarrow 2R\sqrt{R^2 - r^2} = 3r^2 - 2R^2 \end{aligned}$$

$$\begin{aligned} \Rightarrow 4R^2 (R^2 - r^2) &= (3r^2 - 2R^2)^2 \\ \Rightarrow 4R^4 - 4R^2 r^2 &= 9r^4 + 4R^4 - 12r^2 R^2 \\ \Rightarrow 9r^4 &= 8R^2 r^2 \\ \Rightarrow r^2 &= \frac{8}{9} R^2 \end{aligned}$$

$$\text{When } r^2 = \frac{8}{9} R^2, \text{ then } \frac{d^2V}{dr^2} < 0.$$

\therefore By second derivative test, the volume of the cone is the maximum when $r^2 = \frac{8}{9} R^2$.

$$\text{When } r^2 = \frac{8}{9} R^2, \quad h = R + \sqrt{R^2 - \frac{8}{9} R^2} = R + \sqrt{\frac{1}{9} R^2} = R + \frac{R}{3} = \frac{4}{3} R.$$

Therefore,

$$\begin{aligned} &= \frac{1}{3} \pi \left(\frac{8}{9} R^2 \right) \left(\frac{4}{3} R \right) \\ &= \frac{8}{27} \left(\frac{4}{3} \pi R^3 \right) \\ &= \frac{8}{27} \times (\text{Volume of the sphere}) \end{aligned}$$

Hence, the volume of the largest cone that can be inscribed in the sphere is $\frac{8}{27}$ the volume of the sphere.

Q24 : Show that the right circular cone of least curved surface and given volume has an altitude equal to $\sqrt{2}$ time the radius of the base.

Q25 : Show that the semi-vertical angle of the cone of the maximum volume and of given slant height is $\tan^{-1} \sqrt{2}$.

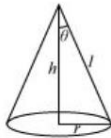
Answer :

Let θ be the semi-vertical angle of the cone.

It is clear that $\theta \in \left[0, \frac{\pi}{2}\right]$.

Let r , h , and l be the radius, height, and the slant height of the cone respectively.

The slant height of the cone is given as constant.



Now, $r = l \sin \theta$ and $h = l \cos \theta$

$$\text{Now, } \frac{dV}{d\theta} = 0$$

$$\Rightarrow \sin^3 \theta = 2 \sin \theta \cos^2 \theta$$

$$\Rightarrow \tan^2 \theta = 2$$

$$\Rightarrow \tan \theta = \sqrt{2}$$

$$\Rightarrow \theta = \tan^{-1} \sqrt{2}$$

Now, when $\theta = \tan^{-1} \sqrt{2}$, then $\tan^2 \theta = 2$ or $\sin^2 \theta = 2 \cos^2 \theta$.

Then, we have:

$$\frac{d^2V}{d\theta^2} = \frac{l^3 \pi}{3} [2 \cos^3 \theta - 14 \cos^3 \theta] = -4\pi l^3 \cos^3 \theta < 0 \text{ for } \theta \in \left[0, \frac{\pi}{2}\right]$$

\therefore By second derivative test, the volume (V) is the maximum when $\theta = \tan^{-1} \sqrt{2}$.

Hence, for a given slant height, the semi-vertical angle of the cone of the maximum volume is $\tan^{-1} \sqrt{2}$.

Now, $r = l \sin \theta$ and $h = l \cos \theta$

The volume (V) of the cone is given by,

$$\begin{aligned}
 V &= \frac{1}{3} \pi r^2 h \\
 &= \frac{1}{3} \pi (l^2 \sin^2 \theta) (l \cos \theta) \\
 &= \frac{1}{3} \pi l^3 \sin^2 \theta \cos \theta \\
 \therefore \frac{dV}{d\theta} &= \frac{l^3 \pi}{3} [\sin^2 \theta (-\sin \theta) + \cos \theta (2 \sin \theta \cos \theta)] \\
 &= \frac{l^3 \pi}{3} [-\sin^3 \theta + 2 \sin \theta \cos^2 \theta] \\
 \frac{d^2V}{d\theta^2} &= \frac{l^3 \pi}{3} [-3 \sin^2 \theta \cos \theta + 2 \cos^3 \theta - 4 \sin^2 \theta \cos \theta] \\
 &= \frac{l^3 \pi}{3} [2 \cos^3 \theta - 7 \sin^2 \theta \cos \theta]
 \end{aligned}$$

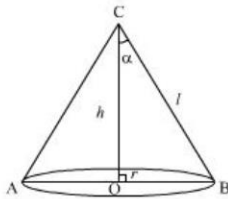
Q26 : Show that semi-vertical angle of right circular cone of given surface area and maximum volume is

$$\sin^{-1}\left(\frac{1}{3}\right).$$

Answer :

Let r be the radius, l be the slant height and h be the height of the cone of given surface area, S .

Also, let α be the semi-vertical angle of the cone.



$$\text{Then } S = \pi r l + \pi r^2$$

$$\Rightarrow l = \frac{S - \pi r^2}{\pi r} \dots (1)$$

Let V be the volume of the cone.

$$\text{Then } V = \frac{1}{3} \pi r^2 h$$

$$\Rightarrow V^2 = \frac{1}{9} \pi^2 r^4 h^2$$

$$= \frac{1}{9} \pi^2 r^4 (l^2 - r^2) [As l^2 = r^2 + h^2]$$

$$= \frac{1}{9} \pi^2 r^4 \left[\left(\frac{S - \pi r^2}{\pi r} \right)^2 - r^2 \right]$$

$$= \frac{1}{9} \pi^2 r^4 \left[\frac{(S - \pi r^2)^2 - \pi^2 r^4}{\pi^2 r^2} \right]$$

$$= \frac{1}{9} r^2 (S^2 - 2S\pi r^2)$$

$$\Rightarrow V^2 = \frac{1}{9} S r^2 (S - 2\pi r^2) \dots (2)$$

Differentiating (2) with respect to r , we get

$$2V \frac{dV}{dr} = \frac{1}{9} S(2Sr - 8\pi r^3)$$

For maximum or minimum, put $\frac{dV}{dr} = 0$

$$\Rightarrow \frac{1}{9} S(2Sr - 8\pi r^3) = 0$$

$$\Rightarrow 2Sr - 8\pi r^3 = 0 \quad (\text{As } S \neq 0)$$

$$\Rightarrow S = 4\pi r^2 \quad (\text{As } r \neq 0)$$

$$\Rightarrow r^2 = \frac{S}{4\pi}$$

Differentiating again with respect to r , we get

$$2V \frac{d^2V}{dr^2} + 2 \left(\frac{dV}{dr} \right)^2 = \frac{1}{9} S(2S - 24\pi r^2)$$

$$\Rightarrow 2V \frac{d^2V}{dr^2} = \frac{1}{9} S \left(2S - 24\pi \times \frac{S}{4\pi} \right) \quad \left(\text{As } \frac{dV}{dr} = 0 \text{ and } r^2 = \frac{S}{4\pi} \right)$$

$$= \frac{1}{9} S(2S - 6S)$$

$$= -\frac{4}{9} S^2 < 0$$

Thus, V is maximum when $S = 4\pi r^2$

Q27 : The point on the curve $x^2 = 2y$ which is nearest to the point $(0, 5)$ is

(A) $(2\sqrt{2}, 4)$ (B) $(2\sqrt{2}, 0)$ (C) $(0, 0)$ (D) $(2, 2)$

Answer :

The given curve is $x^2 = 2y$.

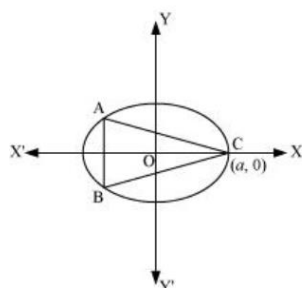
For each value of x , the position of the point will be $\left(x, \frac{x^2}{2} \right)$.

Let P and $A(0, 5)$ are the given points.

Now distance between the points P and A is given by,

∴ Find the maximum area of an isosceles triangle inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with its vertex at one end of the major axis.

Answer :



Let the major axis be along the x -axis.

Let ABC be the triangle inscribed in the ellipse where vertex C is at $(a, 0)$.

Since the ellipse is symmetrical with respect to the x -axis and y -axis, we can assume the coordinates of A to be $(-x_1, y_1)$ and the coordinates of B to be $(-x_1, -y_1)$.

$$\text{Now, we have } y_1 = \pm \frac{b}{a} \sqrt{a^2 - x_1^2}.$$

$$\therefore \text{Coordinates of } A \text{ are } \left(-x_1, \frac{b}{a} \sqrt{a^2 - x_1^2} \right) \text{ and the coordinates of } B \text{ are } \left(-x_1, -\frac{b}{a} \sqrt{a^2 - x_1^2} \right).$$

The given ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Now, we have $y_1 = \pm \frac{b}{a} \sqrt{a^2 - x_1^2}$.

\therefore Coordinates of A are $\left(-x_1, \frac{b}{a} \sqrt{a^2 - x_1^2}\right)$ and the coordinates of B are $\left(x_1, -\frac{b}{a} \sqrt{a^2 - x_1^2}\right)$.

As the point (x_1, y_1) lies on the ellipse, the area of triangle ABC (A) is given by,

$$A = \frac{1}{2} \left| a \left(\frac{2b}{a} \sqrt{a^2 - x_1^2} \right) + (-x_1) \left(-\frac{b}{a} \sqrt{a^2 - x_1^2} \right) + (-x_1) \left(-\frac{b}{a} \sqrt{a^2 - x_1^2} \right) \right|$$

$$\Rightarrow A = b \sqrt{a^2 - x_1^2} + x_1 \frac{b}{a} \sqrt{a^2 - x_1^2} \quad \dots(1)$$

$$\therefore \frac{dA}{dx_1} = \frac{-2x_1 b}{2\sqrt{a^2 - x_1^2}} + \frac{b}{a} \sqrt{a^2 - x_1^2} - \frac{2bx_1^2}{a2\sqrt{a^2 - x_1^2}}$$

$$= \frac{b}{a\sqrt{a^2 - x_1^2}} \left[-x_1 a + (a^2 - x_1^2) - x_1^2 \right]$$

$$= \frac{b(-2x_1^2 - x_1 a + a^2)}{a\sqrt{a^2 - x_1^2}}$$

$$\text{Now, } \frac{dA}{dx_1} = 0$$

$$\Rightarrow -2x_1^2 - x_1 a + a^2 = 0$$

$$\Rightarrow x_1 = \frac{a \pm \sqrt{a^2 - 4(-2)(a^2)}}{2(-2)}$$

$$= \frac{a \pm \sqrt{9a^2}}{-4}$$

$$= \frac{a \pm 3a}{-4}$$

$$\Rightarrow x_1 = -a, \frac{a}{2}$$

But, x_1 cannot be equal to $-a$.

$$\therefore x_1 = \frac{a}{2} \Rightarrow y_1 = \frac{b}{a} \sqrt{a^2 - \frac{a^2}{4}} = \frac{ba}{2a} \sqrt{3} = \frac{\sqrt{3}b}{2}$$

$$\text{Now, } \frac{d^2 A}{dx_1^2} = \frac{b}{a} \left[\frac{\sqrt{a^2 - x_1^2}(-4x_1 - a) - (-2x_1^2 - x_1 a + a^2) \frac{(-2x_1)}{2\sqrt{a^2 - x_1^2}}}{a^2 - x_1^2} \right]$$

$$= \frac{b}{a} \left[\frac{(a^2 - x_1^2)(-4x_1 - a) + x_1(-2x_1^2 - x_1 a + a^2)}{(a^2 - x_1^2)^{\frac{3}{2}}} \right]$$

$$= \frac{b}{a} \left[\frac{2x^3 - 3a^2 x - a^3}{(a^2 - x_1^2)^{\frac{3}{2}}} \right]$$

\therefore Maximum area of the triangle is given by,

$$A = b \sqrt{a^2 - \frac{a^2}{4}} + \left(\frac{a}{2} \right) \frac{b}{a} \sqrt{a^2 - \frac{a^2}{4}}$$

$$= ab \frac{\sqrt{3}}{2} + \left(\frac{a}{2} \right) \frac{b}{a} \times \frac{a\sqrt{3}}{2} = \frac{ab\sqrt{3}}{2} + \frac{ab\sqrt{3}}{4} = \frac{3\sqrt{3}}{4} ab$$

Also, when $x_1 = \frac{a}{2}$, then

$$\frac{d^2 A}{dx_1^2} = \frac{b}{a} \left[\frac{2 \frac{a^3}{8} - 3 \frac{a^3}{2} - a^3}{\left(\frac{3a^2}{4} \right)^{\frac{3}{2}}} \right] = \frac{b}{a} \left[\frac{\frac{a^3}{4} - \frac{3}{2} a^3 - a^3}{\left(\frac{3a^2}{4} \right)^{\frac{3}{2}}} \right]$$

$$= -\frac{b}{a} \left[\frac{\frac{9}{4} a^3}{\left(\frac{3a^2}{4} \right)^{\frac{3}{2}}} \right] < 0$$

Thus, the area is the maximum when $x_1 = \frac{a}{2}$.

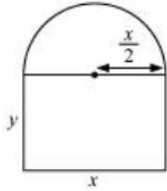
\therefore Maximum area of the triangle is given by,

Q11 : A window is in the form of rectangle surmounted by a semicircular opening. The total perimeter of the window is 10 m. Find the dimensions of the window to admit maximum light through the whole opening.

Answer :

Let x and y be the length and breadth of the rectangular window.

Radius of the semicircular opening = $\frac{x}{2}$



It is given that the perimeter of the window is 10 m.

$$\therefore x + 2y + \frac{\pi x}{2} = 10$$

It is given that the perimeter of the window is 10 m.

$$\therefore x + 2y + \frac{\pi x}{2} = 10$$

$$\Rightarrow x \left(1 + \frac{\pi}{2}\right) + 2y = 10$$

$$\Rightarrow 2y = 10 - x \left(1 + \frac{\pi}{2}\right)$$

$$\Rightarrow y = 5 - x \left(\frac{1 + \pi}{2}\right)$$

\therefore Area of the window (A) is given by,

$$\begin{aligned} A &= xy + \frac{\pi}{2} \left(\frac{x}{2}\right)^2 \\ &= x \left[5 - x \left(\frac{1 + \pi}{2}\right) \right] + \frac{\pi}{8} x^2 \\ &= 5x - x^2 \left(\frac{1 + \pi}{2}\right) + \frac{\pi}{8} x^2 \end{aligned}$$

$$\therefore \frac{dA}{dx} = 5 - 2x \left(\frac{1 + \pi}{2}\right) + \frac{\pi}{4} x$$

$$= 5 - x \left(1 + \frac{\pi}{2}\right) + \frac{\pi}{4} x$$

$$\therefore \frac{d^2A}{dx^2} = - \left(1 + \frac{\pi}{2}\right) + \frac{\pi}{4} = -1 - \frac{\pi}{4}$$

$$\text{Now, } \frac{dA}{dx} = 0$$

$$\Rightarrow 5 - x \left(1 + \frac{\pi}{2}\right) + \frac{\pi}{4} x = 0$$

$$\Rightarrow 5 - x - \frac{\pi}{4} x = 0$$

$$\Rightarrow x \left(1 + \frac{\pi}{4}\right) = 5$$

$$\Rightarrow x = \frac{5}{\left(1 + \frac{\pi}{4}\right)} = \frac{20}{\pi + 4}$$

Thus, when $x = \frac{20}{\pi + 4}$ then $\frac{d^2A}{dx^2} < 0$.

Therefore, by second derivative test, the area is the maximum when length $x = \frac{20}{\pi + 4}$ m.

Now,

$$y = 5 - \frac{20}{\pi + 4} \left(\frac{2 + \pi}{2}\right) = 5 - \frac{5(2 + \pi)}{\pi + 4} = \frac{10}{\pi + 4}$$

Hence, the required dimensions of the window to admit maximum light is given by

$$\text{length} = \frac{20}{\pi + 4} \text{ m and breadth} = \frac{10}{\pi + 4} \text{ m.}$$

Q12 : A point on the hypotenuse of a triangle is at distance a and b from the sides of the triangle.

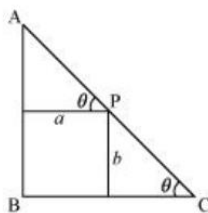
Show that the minimum length of the hypotenuse is $\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}$

Answer :

Let $\triangle ABC$ be right-angled at B. Let $AB = x$ and $BC = y$.

Let P be a point on the hypotenuse of the triangle such that P is at a distance of a and b from the sides AB and BC respectively.

Let $\angle C = \theta$.



We have,

$$AC = \sqrt{x^2 + y^2}$$

Now,

$$PC = b \operatorname{cosec} \theta$$

$$\text{And, } AP = a \sec \theta$$

$$\therefore AC = AP + PC$$

$$\Rightarrow AC = b \operatorname{cosec} \theta + a \sec \theta \dots (1)$$

$$\therefore \frac{d(AC)}{d\theta} = -b \operatorname{cosec} \theta \cot \theta + a \sec \theta \tan \theta$$

$$\therefore \frac{d(AC)}{d\theta} = 0$$

$$\Rightarrow a \sec \theta \tan \theta = b \operatorname{cosec} \theta \cot \theta$$

$$\Rightarrow \frac{a}{\cos \theta} \cdot \frac{\sin \theta}{\cos \theta} = \frac{b}{\sin \theta} \cdot \frac{\cos \theta}{\sin \theta}$$

$$\Rightarrow a \sin^3 \theta = b \cos^3 \theta$$

$$\Rightarrow (a)^{\frac{1}{3}} \sin \theta = (b)^{\frac{1}{3}} \cos \theta$$

$$\Rightarrow \tan \theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}$$

$$\therefore \sin \theta = \frac{(b)^{\frac{1}{3}}}{\sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}} \text{ and } \cos \theta = \frac{(a)^{\frac{1}{3}}}{\sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}} \dots (2)$$

It can be clearly shown that $\frac{d^2(AC)}{d\theta^2} < 0$ when $\tan \theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}$.

Therefore, by second derivative test, the length of the hypotenuse is the maximum when

$$\tan \theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}.$$

Now, when $\tan \theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}$, we have:

Now, when $\tan \theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}$, we have:

$$\begin{aligned} AC &= \frac{b\sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}}{b^{\frac{1}{3}}} + \frac{a\sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}}{a^{\frac{1}{3}}} && \text{[Using (1) and (2)]} \\ &= \sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}} \left(b^{\frac{2}{3}} + a^{\frac{2}{3}}\right) \\ &= \left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}} \end{aligned}$$

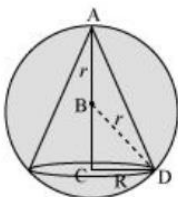
Hence, the maximum length of the hypotenuses is $\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}$.

Q15 : Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$.

Answer :

A sphere of fixed radius (r) is given.

Let R and h be the radius and the height of the cone respectively.



The volume (V) of the cone is given by,

$$V = \frac{1}{3} \pi R^2 h$$

$$= \frac{2\pi r}{3} + \frac{3\sqrt{r^2 - R^2} (2\pi r^2 - 9\pi R^2) + (2\pi Rr^2 - 3\pi R^3)(3R)}{9(r^2 - R^2)} \frac{1}{2\sqrt{r^2 - R^2}}$$

Now, when $R^2 = \frac{8r^2}{9}$, it can be shown that $\frac{d^2V}{dR^2} < 0$.

\therefore The volume is the maximum when $R^2 = \frac{8r^2}{9}$.

When $R^2 = \frac{8r^2}{9}$, height of the cone $= r + \sqrt{r^2 - \frac{8r^2}{9}} = r + \sqrt{\frac{r^2}{9}} = r + \frac{r}{3} = \frac{4r}{3}$.

Hence, it can be seen that the altitude of the right circular cone of maximum volume that inscribed in a sphere of radius r is $\frac{4r}{3}$.

$$= \frac{\pi R r}{3} + \frac{\pi R \sqrt{r^2 - R^2}}{3} - \frac{\pi R^3}{3\sqrt{r^2 - R^2}}$$

$$= \frac{2}{3} \pi R r + \frac{2\pi R (r^2 - R^2) - \pi R^3}{3\sqrt{r^2 - R^2}}$$

$$\text{Now, } \frac{d^2V}{dR^2} = \frac{2\pi r}{3} + \frac{3\sqrt{r^2 - R^2} (2\pi r^2 - 9\pi R^2) - (2\pi Rr^2 - 3\pi R^3)(-6R)}{9(r^2 - R^2)} \frac{1}{2\sqrt{r^2 - R^2}}$$

$$= \frac{2\pi r}{3} + \frac{3\sqrt{r^2 - R^2} (2\pi r^2 - 9\pi R^2) + (2\pi Rr^2 - 3\pi R^3)(3R)}{9(r^2 - R^2)} \frac{1}{2\sqrt{r^2 - R^2}}$$

Now, when $R^2 = \frac{8r^2}{9}$, it can be shown that $\frac{d^2V}{dR^2} < 0$.

\therefore The volume is the maximum when $R^2 = \frac{8r^2}{9}$.

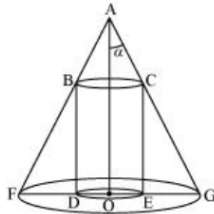
When $R^2 = \frac{8r^2}{9}$, height of the cone $= r + \sqrt{r^2 - \frac{8r^2}{9}} = r + \sqrt{\frac{r^2}{9}} = r + \frac{r}{3} = \frac{4r}{3}$.

Hence, it can be seen that the altitude of the right circular cone of inscribed in a sphere of radius r is $\frac{4r}{3}$.

Q18 : Show that height of the cylinder of greatest volume which can be inscribed in a right circular cone of height h and semi vertical angle α is one-third that of the cone and the greatest volume of cylinder is $\frac{4}{27} \pi h^3 \tan^2 \alpha$.

Answer :

The given right circular cone of fixed height (h) and semi-vertical angle (α) can be drawn as:



Here, a cylinder of radius R and height H is inscribed in the cone.

Then, $\angle GAO = \alpha$, $OG = r$, $OA = h$, $OE = R$, and $CE = H$.

We have

We have,

$$r = h \tan \alpha$$

Now, since $\triangle AOG$ is similar to $\triangle CEG$, we have:

$$\begin{aligned}\frac{AO}{OG} &= \frac{CE}{EG} \\ \Rightarrow \frac{h}{r} &= \frac{H}{r-R} \quad [EG = OG - OE] \\ \Rightarrow H &= \frac{h}{r}(r-R) = \frac{h}{h \tan \alpha}(h \tan \alpha - R) = \frac{1}{\tan \alpha}(h \tan \alpha - R)\end{aligned}$$

Now, the volume (V) of the cylinder is given by,

$$\begin{aligned}V &= \pi R^2 H = \frac{\pi R^2}{\tan \alpha}(h \tan \alpha - R) = \pi R^2 h - \frac{\pi R^3}{\tan \alpha} \\ \therefore \frac{dV}{dR} &= 2\pi R h - \frac{3\pi R^2}{\tan \alpha} \\ \text{Now, } \frac{dV}{dR} &= 0\end{aligned}$$

$$\begin{aligned}\Rightarrow 2\pi R h &= \frac{3\pi R^2}{\tan \alpha} \\ \Rightarrow 2h \tan \alpha &= 3R \\ \Rightarrow R &= \frac{2h}{3} \tan \alpha\end{aligned}$$

$$\text{Now, } \frac{d^2V}{dR^2} = 2\pi h - \frac{6\pi R}{\tan \alpha}$$

And, for $R = \frac{2h}{3} \tan \alpha$, we have:

$$\frac{d^2V}{dR^2} = 2\pi h - \frac{6\pi}{\tan \alpha} \left(\frac{2h}{3} \tan \alpha \right) = 2\pi h - 4\pi h = -2\pi h < 0$$

\therefore By second derivative test, the volume of the cylinder is the greatest when

$$R = \frac{2h}{3} \tan \alpha.$$

$$\text{When } R = \frac{2h}{3} \tan \alpha, H = \frac{1}{\tan \alpha} \left(h \tan \alpha - \frac{2h}{3} \tan \alpha \right) = \frac{1}{\tan \alpha} \left(\frac{h \tan \alpha}{3} \right) = \frac{h}{3}.$$

Thus, the height of the cylinder is one-third the height of the cone when the volume of the cylinder is the greatest.

Now, the maximum volume of the cylinder can be obtained as:

$$\pi \left(\frac{2h}{3} \tan \alpha \right)^2 \left(\frac{h}{3} \right) = \pi \left(\frac{4h^2}{9} \tan^2 \alpha \right) \left(\frac{h}{3} \right) = \frac{4}{27} \pi h^3 \tan^2 \alpha$$

Hence, the given result is proved.

All the best!